HAND IN
Answers recorded
in question paper

QUEEN'S UNIVERSITY
FACULTY OF ARTS AND SCIENCE
SCHOOL OF COMPUTING

CISC-203*
DISCRETE MATHEMATICS FOR COMPUTING SCIENCE

TEST 3
November 2005

Professor Selim G. AKL

Please write your answer to each question only in the box marked **Answer**.
No questions will be answered by the instructor during the exam.
This is a closed-book exam. No computers or calculators are allowed.
If you are unsure of what is wanted for a particular question,
make a reasonable assumption and write this at the beginning of your answer.
PLEASE NOTE: Proctors are unable to respond to queries about the interpretation
of exam questions. Do your best to answer exam questions as written.

NAME: ____

STUDENT NUMBER: ____________________________

FOR INSTRUCTOR'S USE ONLY

Question 1: ______ / 5

Question 2: ______ / 5

Question 3: ______ / 5

Question 4: ______ / 5

TOTAL: _________ / 20

Question 1: [5 marks]

A bowl contains ten red balls and ten blue balls. You select balls at random without looking at them (the selected balls are taken out of the bowl and not replaced).

Answer:

(i) How many balls must you select to be sure of having at least three balls of the same color? Justify your answer.

Hint: Use the pigeonhole principle.

We use the pigeonhole principle where the two colors are the pigeonholes. If 5 balls are selected, at least $\lceil 5/2 \rceil = 3$ must have the same color.

(ii) How many balls must you select to be sure of having at least three blue balls? Justify your answer.

You must select 13 balls to insure at least three blue ones (the first 10 may be red).

Question 2: [5 marks]

Answer the following questions, justifying your answer in each case. How many strings of six lowercase letters of the 26-letter English alphabet contain

Answer:

(i) the letter a, at least once?
$26^6 - 25^6$ = 308,915,776 - 244,140,625 = 64,775,151.

(ii) the letters a and b, each at least once?
$26^6 - (25^6 + 25^6 - 24^6)$ = 11,737,502.

(iii) the letters a and b in consecutive positions, such that a immediately precedes b, and all the letters of the string (including a and b) are distinct?
$5 \times P(24, 4)$ = 1,275,120.

(iv) the letters a and b, where a is somewhere to the left of b in the string, and all the letters of the string (including a and b) are distinct?
$C(6, 2) \times P(24, 4)$ = 3,825,360.

Question 3: [5 marks]

(a) Let n, r, and k be nonnegative integers, where $r \leq n$ and $k \leq r$. Prove the identity:

$$C(n,r) \times C(r,k) = C(n,k) \times C(n-k, r-k).$$

Hint: Expand the two sides of the equality.

Answer:

$$C(n,r) \times C(r,k) = \frac{n!}{r!(n-r)!} \times \frac{r!}{k!(r-k)!} = \frac{n!}{k!(n-r)!(r-k)!}$$

$$C(n,k) \times C(n-k, r-k) = \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{(r-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}$$

(b) Let p be a prime and k an integer such that $1 \leq k \leq p-1$. Show that p divides $C(p,k)$.

Hint: Show that p divides the numerator, but not the denominator.

Answer:

$$C(p,k) = \frac{p!}{k!(p-k)!}$$

It is clear that p is a divisor of $p!$. We must now show that p does not divide the denominator (otherwise, the factor p would cancel out when the fraction is reduced to lowest terms). This follows from the fact that all the prime factors of $k!$ and $(p-k)!$ are smaller than p.

Question 4: [5 marks]

(a) How many different ways are there to choose 12 donuts from the 21 varieties at a donut shop? You can assume that the shop has at least 12 donuts of each variety available. Justify your answer.

Answer:

$C(21 + 12 - 1, 12) = C(32, 12) = 225,792,840.$

(b) How many strings of 20 decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s? Justify your answer.

Answer:

$$\frac{20!}{2!4!3!1!2!3!2!3!}$$