HAND IN
Answers recorded
in question paper

QUEEN'S UNIVERSITY
FACULTY OF ARTS AND SCIENCE
SCHOOL OF COMPUTING

CISC-203*
DISCRETE MATHEMATICS FOR COMPUTING SCIENCE

TEST 4
November 2005

Professor Selim G. AKL

Please write your answer to each question only in the box marked **Answer**.
No questions will be answered by the instructor during the exam.
This is a closed-book exam. No computers or calculators are allowed.
If you are unsure of what is wanted for a particular question,
make a reasonable assumption and write this at the beginning of your answer.
PLEASE NOTE: Proctors are unable to respond to queries about the interpretation
of exam questions. Do your best to answer exam questions as written.

NAME: _____

STUDENT NUMBER: _____

FOR INSTRUCTOR'S USE ONLY

Question 1: _____ / 5

Question 2: _____ / 5

Question 3: _____ / 5

Question 4: _____ / 5

TOTAL: _____ / 20

Question 1: [5 marks]

(a) Find a recurrence relation for the number of ways to climb n stairs, $n \geq 0$, if the person climbing the stairs can take one, two, or three stairs at a time.

Answer:

Let a_n be the number of ways to climb n stairs. For $n \geq 3$, we have

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

(b) What are the initial conditions?

Answer:

$$a_0 = 1, a_1 = 1, a_2 = 2.$$

(c) How many ways can this person climb a flight of eight stairs?

Answer:

$$a_3 = 4, a_4 = 7, a_5 = 13, a_6 = 24, a_7 = 44, a_8 = 81.$$

Question 2: [5 marks]

Let $n = 2^m$, where $m = 2^k$, for $k \geq 0$. Now suppose that some function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log_2 n$, with $f(2) = 1$.

Answer:

(i) Compute $f(16)$.

$$f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 4f(2) + 4 + 4 = 12.$$

Answer:

(ii) Solve the recurrence relation for $f(n)$.

Since $n = 2^m$, $\sqrt{n} = n^{1/2} = 2^{m/2}$, and $\log_2 n = m$, we have:

$$\begin{aligned}
f(2^m) &= 2f(2^{m/2}) + m \\
&= 2[2f(2^{m/2^2}) + (m/2)] + m \\
&= 2^2 f(2^{m/2^2}) + 2m \\
&= 2^2[2f(2^{m/2^3}) + (m/4)] + 2m \\
&= 2^3 f(2^{m/2^3}) + 3m \\
&= 2^3[2f(2^{m/2^4}) + (m/8)] + 3m \\
&= 2^4 f(2^{m/2^4}) + 4m \\
&\vdots \\
&= 2^k f(2^{m/2^k}) + km \\
&= mf(2) + m\log_2 m \\
&= \log_2 n + \log_2 n \cdot \log_2 \log_2 n.
\end{aligned}$$

Question 3: [5 marks]

(a) List all relations on the set $\{0, 1\}$.

Answer:

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0),(0,1)\}$
7. $\{(0,0),(1,0)\}$
8. $\{(0,0),(1,1)\}$
9. $\{(0,1),(1,0)\}$
10. $\{(0,1),(1,1)\}$
11. $\{(1,0),(1,1)\}$
12. $\{(0,0),(0,1),(1,0)\}$
13. $\{(0,0),(0,1),(1,1)\}$
14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
16. $\{(0,0),(0,1),(1,0),(1,1)\}$

(b) Of the relations listed in part (a), identify

Answer:

(i) two nonempty reflexive relations

The relations 8, 13, 14, 16 are all reflexive.

(ii) two nonempty irreflexive relations

The relations 1, 3, 4, 9 are all irreflexive.

(iii) two nonempty symmetric relations

The relations 1, 2, 5, 8, 9, 12, 15, 16 are all symmetric.

(iv) two nonempty antisymmetric relations

The relations 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14 are all antisymmetric.

(v) two nonempty asymmetric relations

The relations 1, 3, 4 are all asymmetric.

(vi) two nonempty transitive relations

The relations 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16 are all transitive.

Question 4: [5 marks]

(a) Let R be a relation on a set of cities $\{a, b, c, \ldots\}$. The pair (a, b) is in R if and only if there is a direct non-stop airline flight from city a to city b. When is (a, b) in

Answer:
(i) R^2?
The pair (a, b) is in R^2 if and only if there is a city c such that the pairs (a, c) and (c, b) are both in R.
(ii) R^3?
The pair (a, b) is in R^3 if and only if there are cities c and d such that the pairs (a, c), (c, d), and (d, b) are all in R.
(iii) R^*?
The pair (a, b) is in R^* if it is possible to fly from a to b.

(b) If a relation R is reflexive, is R^* necessarily reflexive? Explain your answer.

Answer:

Since $R \subseteq R^*$, then clearly whenever $(a, a) \in R$ it is also the case that $(a, a) \in R^*$. If R is reflexive, then so is R^*.

(c) If a relation R is irreflexive, is R^2 necessarily irreflexive? Explain your answer.

Answer:

Not necessarily. For example, if $R = \{(a, b), (b, a)\}$, then R^2 will contain the pair (a, a).