HAND IN
Answers recorded
in question paper

QUEEN'S UNIVERSITY
FACULTY OF ARTS AND SCIENCE
SCHOOL OF COMPUTING

CISC-203*
DISCRETE MATHEMATICS FOR COMPUTING SCIENCE

TEST 5
November 2006

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Please write your answer to each question only in the box marked **Answer**.
No questions will be answered by the instructor during the exam.
This is a closed-book exam. No computers or calculators are allowed.
If you are unsure of what is wanted for a particular question,
make a reasonable assumption and write this at the beginning of your answer.
PLEASE NOTE: Proctors are unable to respond to queries about the interpretation
of exam questions. Do your best to answer exam questions as written.

NAME: ____

STUDENT NUMBER: ____________________________

FOR INSTRUCTOR'S USE ONLY

Question 1: ______ / 5

Question 2: ______ / 5

Question 3: ______ / 5

Question 4: ______ / 5

TOTAL: _________ / 20

Question 1: [5 marks]

a) Show that the relation

$$D = \{(x, y) \mid x - y \text{ is an integer } \}$$

is an equivalence relation on the set of real numbers.

Answer:

- The relation is reflexive since $x - x$ is an integer, thus $(x, x) \in D$ for all x.
- The relation is symmetric since, if $(x, y) \in D$ then $x - y$ is an integer, say z. Therefore $y - x$ is integer $-z$. Thus $(y, x) \in D$.
- Suppose $(x, y) \in D$ and $(y, z) \in D$. Then $x - y = z_1$ and $y - z = z_2$ for integers z_1 and z_2. Therefore $(x-y)+(y-z) = x-z = z_1+z_2$. Thus $x-z$ is an integer and $(x, z) \in D$. Therefore the relation is transitive.

b) Describe the equivalence class for each of 0 and 0.5.

Answer:

The equivalence class for 0 is the set of integers Z (since k is related to 0 if and only if $k - 0$ is an integer).
The equivalence class for .5 is the set $k + .5$ for all integers k (since any arbitrary number x is related to 0.5 if and only if $x - 0.5$ is an integer, and hence x must be of the form $k + 0.5$ for all integers k).

Question 2: [5 marks]

Let R be the relation on the set of ordered pairs of positive integers such that

$$((a,b),(c,d)) \in R \quad \text{if and only if} \quad ad \leq bc.$$

a) Is the relation reflexive? Explain your answer.
Yes, because $((a,b),(a,b)) \in R$ for every (a,b) since $ab \leq ba$.

b) Is it symmetric? Explain your answer.
No, because $((1,3),(2,4)) \in R$ but $((2,4),(1,3)) \notin R$.

c) Is it antisymmetric? Explain your answer.
No, because $((3,2),(6,4)) \in R$ and $((6,4),(3,2)) \in R$.

d) Is it transitive? Explain your answer.
Yes, because if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$, then $ad \leq bc$ and $cf \leq de$, implying that $af \leq be$, and therefore $((a,b),(e,f)) \in R$.

e) Is it an equivalence relation? Explain your answer.
No, since it is not symmetric.

f) Is it a partial order? Explain your answer.
No, since it is not antisymmetric.

Question 3: [5 marks]

It is required to determine whether the following two graphs, given by their adjacency matrices,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

are isomorphic. If they are isomorphic you are to exhibit an isomorphism. If they are not, give a rigorous argument that none exists.

Hint: You may find it helpful to draw the two graphs.

Answer:
The two graphs are not isomorphic since in the first graph the two vertices of degree 3 (see rows 3 and 6) are adjacent to a common vertex (see column 5), while this is not true in the second graph (see rows 2 and 6).

Alternatively, the two graphs are not isomorphic since in the second graph each vertex of degree 3 is adjacent to two vertices of degree 1 and to one vertex of degree 2, while this is not true in the first graph.

Question 4: [5 marks]

Use Huffman coding to encode these symbols with given frequencies: A: 0.23, B: 0.37, C: 0.06, D: 0.14, E: 0.16, F: 0.01, G: 0.03. Compute the average number of bits required to encode a symbol.

Answer:
In what follows, the heavier subtree is always to the left (edges going left are labeled with a 0, while edges going right are labeled with a 1). Combine F and G into a tree T_1 of weight 0.04. Combine T_1 and C into a tree T_2 of weight 0.10. Combine T_2 and D into a tree T_3 of weight 0.24. Combine E and A into a tree T_4 of weight 0.39. Combine B and T_3 into a tree T_5 of weight 0.61. Finally, combine T_4 and T_5. This gives the following code: A: 10, B: 00, C: 0110, D: 010, E: 11, F: 01111, G: 01110.
The average number of bits required is given by:

$$2(0.23) + 2(0.37) + 4(0.06) + 3(0.14) + 2(0.16) + 5(0.01) + 5(0.03) = 2.38$$

which means that, on average, 2.38 bits are needed to encode a character.