CISC 204 Class 6

Proof Rules for the Law of Excluded Middle and Disjunction Elimination

Text Correspondence: pp. 16–20, 25–26

Main Concepts:

- *Proof using contradiction: a rule that uses an assumption box and a contradiction*
- *Law of Excluded Middle (LEM): a derived rule of proof*
- $\vee$ *e: disjunction elimination*

This class explores two proofs rule that uses a mode of reasoning about "either…or". The first rule is that, for any proposition whatsoever, either it is true or its negation is true. THe second rule is that, if at least one of two propostions is true and we can use *each* of them to conclude the same third proposition, then the third proposition is true.

6.1 Law of Excluded Middle, or LEM

Proof Rule, derived: *Law of Excluded Middle*, LEM

$$\frac{}{\phi \vee \neg \phi}\ \text{LEM}$$

The Law of Excluded Middle states that, at any line in a proof, we can introduce the disjunction of a proposition and its negation. This is a very powerful rule, especially when it is combined with the rules of disjunction-elimination and proof by contradiction.

6.2 Disjunction Elimination

Suppose that we have a proposition that is a simple disjunction such as $p \vee q$. If this proposition is a line in a valid proof in natural deduction, then at least one of p and q is taken to be true. We can contruct "paralle" lines of reasoning, first assuming p and then assuming q. If we can deduce the same "goal" formula from p and from q, then the goal formula is true and can be written as a new line in our valid proof.

When we perform "either…or" reasoning, we are eliminating the disjunction and deducing another formula. If the disjunction os $\phi \vee \psi$, and the goal formula is χ, then we need two assumption boxes that have the same goal formula as the line line of each box.

Proof Rule: **disjunction**-*elimination*: $\vee$e

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\text{e}$$

We can now prove that disjunction is commutative:

$$p \vee q \vdash q \vee p$$ *H&R, p. 17*

We can also prove that disjunction can be introduced *into* an implication:

$$q \to r \vdash p \vee q \to p \vee r$$ *H&R, p. 18*

Here is a sequent for self-study. An example proof strategy is on the next page of these notes.

Self-Study Sequent 6.1:

"Suppose both p, and either q or r, is true. We can conclude that either p and q is true, or that p and r is true"

$$p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$$

Self-Study Sequent 6.1:

Proof Strategy:

We see that the premise is a conjunction. This suggests that we can use $\wedge$e to determine that each of the conjuncts is true. The first few steps can be

1	$p \wedge (q \vee r)$	premise
2	p	$\wedge$ e$_1$ 1
3	$q \vee r$	$\wedge$ e$_2$ 1

Looking back at the original problem, we see that the conclusion of the sequent is a disjunction. Although it does not completely eliminate other strategies, it suggests that we should consider using one of the disjunction rules as our final step. In particular, we should look closely at the new rule $\vee$e that would let us independently assume q and r; if, from each of these, we could conclude $((p \wedge q) \vee (p \wedge r))$ then our proof would be complete.

This works! From q we can conclude $p \wedge q$ and so on to complete the proof.