

CISC 204 Class 8

Correctness and Completeness of Propositional Logic

Text Correspondence: pp. 45–53

Main Concepts:

- *Tautology: a formula that is always true*
- *Correct theory: every theorem is a tautology*
- *Complete theory: every tautology is a theorem*

This class is about *meta-logic*, which means that we are reasoning about a logical system. The object of study is the entire axiomatic system of propositional logic that is presented in the text. We want to know some fairly basic things about any logical system, such as:

1. Is there a relationship between valid sequents and truth tables?
2. Is every theorem always true?
3. Is every formula that is always true always provable?

We will address item (1) with a definition, allowing us to relate syntax and semantics. Item (2) is known as *soundness* or *correctness*. Item (3) is known as *completeness*.

As we saw in the last class, the truth of a propositional formula can be determined using truth tables. We wish to show that the proof rules are “correct”, by which we mean that valid sequents preserve truth. This correctness comes from computing a truth table, which is a semantic model.

For this class and the next class, we will use a common abbreviation that is not in the text. We will use the capital Greek symbol Γ to represent a set of formulas. For our purposes, we can say that Γ has n formulas or that

$$\Gamma = \{\phi_1, \phi_2, \dots, \phi_n\}$$

To begin, suppose that we have a sequent that may have any number of premises and propositions that are derived using the rules of inference. The conclusion is valid if the sequent

$$\Gamma \vdash \psi$$

is valid. This is a *syntactic* determination, because it depends only on the well formed formulas $\phi_i \in \Gamma$ and on the rules that allow deduction of a formula from other formulas.

The corresponding *semantic* determination is using a model, the fullest model being a truth table.

Definition: Semantic entailment

If, for all valuations in which all $\phi_i \in \Gamma$ evaluate to \mathbf{T} , ψ also evaluates to \mathbf{T} , we say that

$$\Gamma \models \psi$$

holds and we call \models the *semantic entailment* relation.

Another way of talking about such a sequent is to say that the formulas $\phi_i \in \Gamma$ *semantically entail* the conclusion. Note that we are careful not to use syntactic ideas or terms, such as “proves”, because we do not yet know for sure what the full relations are.

In exploring the definition, we can see that it specifically confines semantic entailment to models in which *every* formula $\phi_i \in \Gamma$ evaluates to, or is assigned, the truth value \mathbf{T} . This begins with the premise formulas, where we do not consider models with a false premise. It continues with deduced formulas, which from the rules of deduction we fully expect to be true if the premises are true.

In class we discussed the examples from the text:

1. Does $p \wedge q \models p$?
2. Does $p \vee q \models p$?
3. Does $p \vee q, \neg p \vee q \models q$?
4. Does $p \models q \vee \neg q$?

Example (4) is an excellent illustration of the difference between a valid sequent, a valid proof, and semantic entailment. It is a valid sequent although the proof is incomplete; the semantic entailment holds.

In the last class, we emphasized that a truth table is not a proof. We can now make a clearer assertion: showing that a semantic entailment holds is not sufficient to prove that the consequent follows from the premises.

We also noted that a truth table is useful for determining whether or not to go to the effort of proving that a sequent is valid. We can now say this: if there is a truth assignment such that the premises are all true so they have the value \mathbf{T} , and the consequent under the same truth assignment is false with the value \mathbf{F} , then no valid proof exists. From this class, Example (2) shows that there is a model for which the premises are true and the conclusion is false, so there is no proof for its sequent.

Theorem, Propositional Logic is Correct:

Let $\phi_i \in \Gamma = \{\phi_1, \phi_2, \dots, \phi_n\}$ and ψ be formulas in propositional logic.

if $\Gamma \vdash \psi$ is valid then $\Gamma \models \psi$ holds

Proof: by mathematical induction on the length of the proof. Strictly, we would have to show that every rule of deduction produces a semantic entailment that holds, then reason over the length of the proof.

We have not studied mathematical induction because, to do so properly, we would need to introduce the natural numbers into a logical system and doing so requires more axioms. For the purposes of understanding the soundness of propositional logic, students can use their current understanding of how mathematical induction works. The text has a reasonably good description of the proof; the instructor prefers the more rigorous proof in the text by Jean Rubin.

This theorem says that propositional logic correct. Another way of saying it is that, for every valid proof, true premises *always* lead to a true conclusion, or a semantic entailment that holds. This is reassuring because we now can trust the rules of inference.

We now know that propositional logic is correct, which means that every valid sequent is also a semantic entailment that holds. In symbols, we know that

$$\text{if } \Gamma \vdash \psi \text{ then } \Gamma \models \psi$$

A natural question whether the converse is true, which would mean that every entailment that holds has a valid proof. (This was demonstrated in 1930 by Kurt Gödel in his doctoral dissertation.) To appreciate the proof, it is helpful to introduce a new term.

Definition: Tautology

A formula ψ is called a *tautology* if and only if ψ evaluates to **T** under all possible valuations that is, if and only if $\models \psi$.

Which of the following are tautologies?

$$p \vee \neg p$$

$$p \rightarrow \neg p$$

$$(p \rightarrow q) \rightarrow \neg p \vee q$$

For propositional logic to be *complete*, every semantic entailment would have a corresponding theorem. Tautologies are an important subclass of sequents and semantic entailments because, if ϕ is a tautology, then

$$\vdash \phi \text{ and } \models \phi$$

We can understand that the validity comes from the Law of Excluded Middle, which is one of our derived rules. The semantic entailment comes from a truth table that can be verified for any given tautology.

We can now state Gödel's Completeness Theorem. The proof is for a more advanced class, so we in class we limited ourselves to sketching the outlines of the proof.

Completeness of Propositional Logic:

For a set of formulas $\Gamma = \{\phi_1, \phi_2, \dots, \phi_n\}$ and a formula ψ ,

$$\text{If } \Gamma \models \psi \text{ then } \Gamma \vdash \psi$$

Proof of Completeness: Concepts

The proof of the theorem begins by assuming the antecedent, so we assume that the semantic entailment

$$\phi_1, \phi_2, \dots, \phi_n \models \psi$$

holds. This can be done in three steps:

1. Show that the entailment $\Gamma \models \psi$ implies an entailment of nested related implications. That is, show that the premises Γ and the conclusion ψ constitute a tautology.
2. Show that the nested implications constitute a valid theorem (this is the hard step).
3. Show that the nested implications lead to the consequent of the theorem.

The key to this theorem is converting a set of premises into nested implications. We can see how this works with two simple examples, from which we can generalize a pattern.

Consider the case where the entailment $\phi \models \psi$ holds. Because this holds if and only if the premise is true (ϕ is **T**) and the conclusion is true (ψ is **T**), we can easily establish that

$$\models \phi \rightarrow \psi$$

Consider the case where the entailment $\phi_1, \phi_2 \models \psi$ holds. This entailment holds if and only if all of $\{\phi_1, \phi_2, \psi\}$ are true. From this entailment, we can verify that

$$\models \phi_1 \rightarrow (\phi_2 \rightarrow \psi)$$

which we often wrote, dropping the parentheses, as

$$\models \phi_1 \rightarrow \phi_2 \rightarrow \psi$$

This is one way that we could approach the proof of Step 1 of the Completeness Theorem.

Step 2 is much harder and is more interesting. A preliminary theorem is often used, the proof of which is beyond the scope of this course.

In words, if a formula χ is a tautology then χ is a theorem. In symbols,

Theorem:

If $\models \chi$ holds then $\vdash \chi$ is valid

The concept often used to prove this theorem is expansion of the truth table that is based on the k atomic propositions “inside” χ . As with the Correctness Theorem, mathematical induction is a useful rule that we have not covered fully.

From this theorem, the entailment as a tautology leads directly to the validity of the nested implications.

Step 3 converts the antecedents of the nested implications into premises, leaving the consequent as the conclusion. This can be done by taking all of the formulas ϕ_i as premises, then performing n applications of the rule of implication elimination $\rightarrow e$ to get the results.

In symbols, we can now write a sketch of the proof of the Completeness Theorem. Assume the semantic entailment $\phi_1, \phi_2, \dots, \phi_n \models \psi$ and then:

1. Show that $\models \phi_1 \rightarrow (\phi_2 \rightarrow (\dots(\phi_n \rightarrow \psi)\dots))$ holds.
2. Show that $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\dots(\phi_n \rightarrow \psi)\dots))$ is valid.
3. Show that $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ is valid.