CISC 204 Class 19

Working With Quantifiers

Text Correspondence: pp. 117–119

Main Concepts:

- *Universal quantifiers distribute over conjunction*

The primary purpose of this class will be to improve our ability to prove the validity of sequents in predicate logic. The secondary purpose is to learn important equivalences of quantified formulas; these equivalences will be useful in courses for which this course is a prerequisite.

Example

The first equivalence we will prove is from the textbook, Theorem 2.13, Part 3(a). In English, the first "direction" of the equivalence states: if ϕ holds for all x, and ψ holds for all x, then $(\phi \wedge \psi)$ holds for all x. In symbols, the sequent is

$$\forall x\, \phi \wedge \forall x\, \psi \vdash \forall x\, (\phi \wedge \psi)$$

To avoid the technical matters associated with substituting "t that is free for x" in the formulas, we will prove an instance of the general case that is

$$\forall x\, P(x) \wedge \forall x\, Q(x) \vdash \forall x\, (P(x) \wedge Q(x)) \tag{19.1}$$

After writing the premises as the first line of our proof, and extracting the conjuncts as separate lines, we examine the conclusion. Our strategy is that, because the conclusion is a universal quantifier, a possible rule to use is Universal Introduction $\forall x$ i. To do this we will need to introduce – which means assume the use of – a new variable which we will call z. Our strategy so far will look like

1	$\forall\, x\, P(x) \wedge \forall\, x\, Q(x)$	premise
2	$\forall\, x\, P(x)$	$\wedge$ e$_1$ 1
3	$\forall x\, Q(x)$	$\wedge$ e$_2$ 1
4	z	
	$\vdots$	
	$P(z) \wedge Q(z)$	
?	$\forall x\, (P(x) \wedge Q(x))$	$\forall x$ i 4–?

We know that the last line of the $\forall x$ i box must be the formula we want to conclude, with the substitution of z for x; this means that the last line must be

$$(P(x) \wedge Q(x))[z/x] \qquad \text{which is} \qquad (P(z) \wedge Q(z))$$

How can we deduce such a formula? Going back to our first premise, we can substitute any symbol for x that satisfies the requirements of substitution; naturally, within the box, we would like the substitution to be

$$\phi[z/x] \qquad \text{which is} \qquad P(z)$$

and similarly for the formula in the second premise.

Putting these observations together, one proof of Sequent 19.1 is

1	$\forall x\, P(x) \wedge \forall x\, Q(x)$	premise
2	$\forall x\, P(x)$	$\wedge$ e_1 1
3	$\forall x\, Q(x)$	$\wedge$ e_2 1
4	z	
5	$P(z)$	$\forall x$ e 2
6	$Q(z)$	$\forall x$ e 3
7	$P(z) \wedge Q(z)$	$\wedge$ i 5, 6
9	$\forall x\, (P(x) \wedge Q(x))$	$\forall x$ i 4–8

(19.2)

A similar strategy can be used to prove the second "direction" of the sequent, which is

$$\forall x\, (\phi \wedge \psi) \vdash \forall x\, \phi \wedge \forall x\, \psi \tag{19.3}$$

After writing the premise as Line 1, we would begin an assumption box with a new variable; we could continue to use the symbol z. We could then use the same formula as Proof 19.2, Line 8, which substitutes z for x in $(\phi \wedge \psi)$ in the premise; then we could use the identity of Line 7, and so on.

The remainder of this proof is an excellent exercise for students.