CISC 204 Class 22

Basic Semantics of Predicate Logic

Text Correspondence: pp. 122–123

Main Concepts:

- A*: universe of discourse*
- *Extensional definition: a predicate as a set*
- $\mathcal{P}$*: set of predicate symbols*

We can think of the syntax of a formal language as an answer to the question, "what are we allowed to write?" We can think of the rules of deduction, variously, as answers to a question:

- From premises, how can we deduce a conclusion?
- How can we transform the premise sentences into a conclusion?

The second way is very important to a computer scientist because a great deal of what we do is transforming one representation into another.

22.1 Semantics of Predicate Logic

Semantics are an answer to the simple question, "does this make sense?" By this we mean that, if we assign meanings to variables and functions and predicates, does a proof preserve the truth properties through the allowed syntactic transformations?

To explore the semantics of predicate logic, let us recall the semantics of propositional logic and how we extended its syntax.

The foundation of propositional semantics was that every "atom" had the value **T** or the value **F**. From this, we introduced a model, which was an assignment of truth values to the atoms. We expressed a model, primarily, as a truth table. We discovered two critically important properties of a truth table: a proposition was valid when *every* row of the truth table was **T**, and a proposition was satisfiable when *some* row of the truth table was **T**. (We really referred to a model rather than to a row of a truth table, but for now the truth table will suffice.)

An immediate difficulty we spotted, when a proposition had k atoms, was that the truth table had 2^k rows. It grew exponentially, leaving us with an uneasy feeling about evaluating validity and satisfiability of large propositions. We will have to face this uneasiness directly now, because when we try to think of a model in predicate logic we have to manage the possibility that a variable can take on one of an arbitrarily large number of values.

Our semantics begins with a need to restrict the values that a variable or function can have. Many texts call this the *universe of discourse*, which is the set of values that we are talking about; this course's textbook uses similar language, calling it the universe of concrete values. In an axiomatic system we must take this concept as intuitively understood.

We will refer to the universe of discourse as the set A. Following Georg Cantor, we will think of a **set** as a collection of "clearly defined, distinguishable objects".[1]

A variable, such as x or y, will be restricted to taking on a value from the universe of discourse. In mathematical notation, we would write

$$x \in A$$

Next, we need to understand the concept of a *mapping*, which we will abbreviate as a *map*. This "takes" its inputs "to" its outputs. Here, a map will always have a single determinable result for any given inputs.

An important concept is that a predicate P can be either a mapping or a set. As a mapping, a predicate takes one or more input values and returns either $\mathbf{T}$ or $\mathbf{F}$. In mathematical notation, we would write the mapping concept as

$$\underline{P} : A \to \{\mathbf{T}\ ,\ \mathbf{F}\}$$

We would write the set concept as

$$x \in P$$

The set of all predicate symbols will be written as $\mathcal{P}$, so a predicate P must satisfy the set equation $P \in \mathcal{P}$.

[1]Cantor wrote in German with, surprisingly, a definitive citation not yet clear. One translation is from a footnote on Page 679 of *Mathematical Statistics for Economics and Business*, by R. Mittelhammer (1995) which is: "By set we mean any collection M of clearly defined, distinguishable objects m (which will be called elements of M) which from our perspective or through our reasoning we understand to be a whole".

We can now consider some simple examples of a set A and predicates P and Q. Suppose we restrict our values to be the integers from 0 to 3; we would then have

$$A = \{0, 1, 2, 3\}$$

We might define $P(x)$ to be "x is even" and $Q(x)$ to be "x is odd". With this specific universe A, and these predicates P and Q, we can intuitively see that certain assertions are always true:

- $\exists x\, P(x)$
- $\exists x\, Q(x)$
- $\forall x\, (P(x) \vee Q(x))$

It is crucially important that we understand that these assertions are not *tautologies*, because we cannot prove them as theorems in predicate logic; but they are definitely true for the specific set A and meanings P and Q that we just used. To extend the concepts from propositional logic: these turn out to not be valid formulas, but they are satisfiable.

Extensional Definitions of Predicates

The definitions of predicates that we have provided so far in this course are *intensional*, where we have described the necessary and sufficient conditions for a predicate to be true.

An equally legitimate definition of a predicate is *extensional*, where we enumerate or otherwise specify the arguments for which the predicate is true. This is especially useful for small finite sets, but can be used for larger sets. The choice of an intensional or extensional definition will depend on circumstances. For the semantics of predicate logic, we will always prefer the extensional definition.

In the above example of base-4 arithmetic, we used an intensional definition of the predicates $P^{\mathcal{M}}$ and $Q^{\mathcal{M}}$. The English description that followed the definition provided an intensional definition of each predicate.

In the base-4 example, the universe of discourse was the set $A = \{0, 1, 2, 3\}$. Rather than defining $P(x)$ as "x is even" and $Q(x)$ as "x is odd", we can define the predicates extensionally as

$$\begin{aligned} P &= \{0, 2\} \\ Q &= \{1, 3\} \end{aligned}$$

Relations, which are binary predicates or predicates of two variables, can also be extensionally defined. For small finite sets, the extensional definition is often preferred because it is concise and easy to verify. For example, in base-4 arithmetic, we could intensionally define $R(x, y)$ as "x is not equal to y". An extensional definition would specify the ordered pairs of (x, y) values over which the relations holds; for $x \neq y$ in base-4 arithmetic we could specify

$$R = \{(0, 1), (1, 0)\}$$

A prudent student will take care to understand extensional definitions because the textbook uses them often and they are a clear way to answer test questions that ask for counter-examples to predicate sequents.

End of Extra Notes