CISC 204   Class 23

Semantics of Functions

Text Correspondence: pp. 123–124

Main Concepts:

- *f: function maps values to values*
- *$\mathcal{F}$: set of function symbols*

The next concept we need is that of a function.

23.1 Semantics of a Function

Syntactically, we simply accepted that a function "takes" zero or more terms as input and "returns" a value that a variable can take. We now need to be more careful and specific.

With reference to a set A, we want a nullary (or constant) function to always return a value that is in the set A. In mathematical notation, we would write

$$f : \{\} \rightarrow A$$

This notation says that, taking no inputs, the function f maps to the set A; this is the same as saying that it returns a value in the set A.

Now, suppose that f is a function that maps a single value to one definite value. In mathematical notation, we would write

$$f : A \rightarrow A$$

Next, suppose that f maps two values to one definite value. We write this as $f(x, y)$ according to the syntax of propositional logic. Semantically, we mean that $f(\cdot, \cdot)$ maps a pair, or 2-tuple, of values from the set A to a value in the set A. In mathematical notation, there are two ways to write this; one way is to say explicitly that f takes a 2-tuple from the outer-product set $A \times A$, and the other way is to abbreviate the outer-product set as A^2. The instructor's preferred notation, and the textbook notation, are

$$\begin{aligned} & f : A \times A \rightarrow A \\ \text{or} \quad & f : A^2 \rightarrow A \\ \text{equivalently,} \quad z = \; & f(x, y) \Rightarrow z \in A \end{aligned}$$

This notation extends to a function of n variables, also called an n-ary function, as

$$\begin{array}{rl} & f : A \times A \times \cdots \times A \to A \\ \text{or} & f : A^n \to A \end{array}$$

We must use our notation carefully. In symbolic logic, the set of functions that we are using is typically limited; the set is written using the symbol $\mathcal{F}$. When we mean that a function f is in the set of specified functions, we will write

$$f \in \mathcal{F}$$

This is very different than the usual way of writing function! We might be used to saying that the *value returned by* f is in the set A; we do not need to say this, because it is in the definition of the function f. Instead, we are concerned with restricting our attention to one of a limited number of functions.

For example, thinking of the previous set A that is the integers from 0 to 3, we might define the function s to be the "succeeding" integer in base-4 arithmetic. This is often written as modular arithmetic; we could define the function s in many ways, such as

$$\begin{aligned} s(x) &\stackrel{\text{def}}{=} (x+1)\ \textit{mod}\ 4 \\ s(x) &\stackrel{\text{def}}{=} \begin{cases} x+1 & \text{if} \quad x \leq 2 \\ 0 & \text{if} \quad x = 3 \end{cases} \end{aligned}$$

We might define the function d to be "double" the integer in base-4 arithmetic, so

$$d(x) \stackrel{\text{def}}{=} (2 \cdot x)\ \textit{mod}\ 4$$

Our set, or space, of functions would then be $\mathcal{F} = \{s, d\}$.