

# CISC 204 Class 33

## Undecidability in Predicate Logic

Text Correspondence: pp. 131–136

*Main Concepts:*

- *Decidable logical theory: there is an algorithm to determine whether a given formula is a theorem*

### 33.1 Undecidability of Predicate Logic

Our exploration of semantic entailment introduced two theorems for predicate logic. The Soundness Theorem states that, if there is a proof for a theorem, then the theorem's formula is valid; another way to say this is that predicate logic is *sound* because every theorem is a semantic entailment.

The Completeness Theorem states that, if a formula is valid, then the formula is a theorem. This says that predicate logic is *complete* because we can prove every valid formula. The two theorems are often combined into a single assertion:

For a formula  $\phi$ ,  $\vdash \phi$  if and only if  $\models \phi$

It would seem that this is straightforward: a theorem is semantically entailed, and vice versa. But we must walk carefully here because predicate logic can be subtle.

In propositional logic, we had a mechanical way of determining whether a formula was valid: we could enumerate every possible valuation, or equivalently examine every row in a truth table, to determine whether or not the formula evaluated to **T** in every valuation. In predicate logic we cannot do this, because for a model with an infinitely large universe of discourse  $A$  there is no way to enumerate all possible logical environments.

The Completeness Theorem is a conditional statement that says, *if* a formula is valid, *then* it is a theorem. To apply this theorem we need an answer to the question

For a formula  $\phi$ , is  $\phi$  valid? That is, does  $\models \phi$  hold for all possible models?

This question was asked in 1928 by the great mathematician David Hilbert. Another way to pose this problem, which comes from the equivalence provided by the Soundness Theorem and the Completeness Theorem, is

For a formula  $\phi$ , is  $\phi$  a theorem in predicate logic? That is, is  $\vdash \phi$  a theorem in predicate logic?

The first question is semantic and the second question is syntactic. The answer to these questions is the *Undecidability Theorem*:

*NO: there is no algorithm for determining whether  
a given formula is a theorem in predicate logic*

This result was proved, independently, in 1936 by Alonzo Church and Alan Turing. This result puts a fundamental limit on computation and on expressibility in languages that have predicate logic as a universe of discourse.

When we reflect on this statement, it is perhaps a bit astonishing. In computer science, we are used to being able to find an algorithm to solve the problems given to use. In those cases when we are having difficulty in finding an algorithm we simply try harder. The Undecidability Theorem says that there are fundamental limits to algorithms for solving problems.

We now accept that we cannot prove that a formula  $\phi$  is valid, meaning that we cannot prove that it holds in all models  $\mathcal{M}$ . We might ask whether  $\phi$  is satisfiable, which is asking if there is some model  $\mathcal{M}$  and some environment  $l$  of the model in which  $\phi$  holds. Formally, we are asking

For a formula  $\phi$ , is there a model  $\mathcal{M}$  and environment  $l$  such that  $\models_l \phi$ ?

The answer to this question is also negative. We can see this in a proof by contradiction: suppose there is no environment in which  $\models \phi$  holds. This is semantically equivalent to saying that, for all environments,  $\models \neg \phi$  holds. Because  $\models \neg \phi$  is undecidable, the semantically equivalent statement that  $\models \phi$  is satisfiable is also undecidable. We can summarize this finding as

*NO: there is no algorithm for determining whether  
a given formula in predicate logic is satisfiable*

What happens if we extend predicate logic to include integer arithmetic? This is usually done by adding the Peano Axioms, either as 5 axioms or as 4 axioms and the rule of mathematical induction. What happens is that the above undecidability results continue to hold, plus we get Gödel's Incompleteness Theorem:

*A logical system that includes the Peano Axioms cannot be both  
consistent and complete*

Because we usually insist that a logical system is consistent, meaning that it does not have a contradiction as a theorem, we must accept that for such a system there are valid formulas that are unprovable. What if we find such a formula, and take it as an axiom? Gödel's Incompleteness

Theorem still applies! We therefore must accept that, by being able to express integer arithmetic, there are an infinite number of valid formulas that cannot be proved.

Proof of these theorems is beyond the scope of this course.

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## Extra Notes

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### UNDECIDABILITY AND THE AXIOM OF CHOICE

An example of a useful formula that is undecidable, given the “usual” axioms of mathematics, is the Axiom of Choice. Conceptually, we would begin by extending arithmetic to be able to rigorously define sets and set theory. Set theory was formalized by Ernst Zermelo and extended by Abraham Fraenkel; the result is usually abbreviated as ZF set theory. The axioms can be translated from symbols into plain English as:

1. Two sets are equal if they have the same members
2. Every non-empty set  $A$  contains a member  $x$  such that  $A$  and  $x$  are disjoint sets
3. For a set  $A$ , and formula  $\phi(x)$  with one free variable  $x$ , there exists a set  $B$  that is all the members of  $A$  that satisfy the formula  $\phi(x)$
4. If  $A$  and  $B$  are sets, then there exists a set  $C$  that has  $A$  and  $B$  as members
5. The union of the members of a set of sets is a set
6. For a definable function  $f$  with a domain set  $A$ , the range of  $f$  is a set  $B$
7. There is a set  $A$  that has infinitely many members
8. For any set  $A$ , there is a set  $B$  that contains every subset of  $A$

These axioms can be expressed in predicate logic that has the operator of set membership.

There is an important axiom that is independent of these set axioms. Depending on how it is written, it is called the Axiom of Choice or the Well-Ordering Theorem. One English translation, from symbolic logic, is

9. For any set  $X$  of non-empty sets, for every set  $A$  in  $X$ , there is a function  $f$  that maps some member of  $A$  into  $A$

The Axiom of Choice is not problematic for a finite set of finite sets; in such a case, for each finite  $A$ , we can “order” its members and map the first member to itself as  $f(x) = x$ . We could also take the last member, or use any computable formula to choose from  $A$ .

Also unproblematic are an infinite number of finite sets, where we can order each  $A$  and choose a member, or a finite number of infinite sets, where we can specify a “choice” function  $f$  for each set  $A$  that maps some member of  $A$  into  $A$ .

The problem arises when there is an infinite number of infinite sets. We cannot list all of the choice functions, and we cannot specify a choice function that works for all sets. To some students the Axiom of Choice might seem intuitively evident, but other students might object that they want a construction of the choice functions  $f$  for each set. Because the Axiom of Choice is independent of the other axioms of set theory, each student has a point: we can add this axiom, or its negation for the infinite cases, and have a consistent new set theory.

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End of Extra Notes

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# **CISC 204 Class 34**

## **Advanced Modeling of Logical Systems**

Text Correspondence: ~

The material for this class is presented in a video lecture that accompanies these notes.