CISC271
Fall 2006
Homework for week 9
in preparation for quiz 4
Solutions

The following questions are from Recktenwald Chapter 11.

Questions 15, 20, and 22.

11-15 Here's my m-file

```
function A15 = Q15DR(s)
% This m-file answers question 15 of Chapter 11 (p. 670) in Recktenwald
% It uses gaussquad to evaluate the integrals
% the function to evaluate is
% 1/sqrt(2*pi) * I
% where I is the integral of exp((-z^2)/2) in the interval -s .. s
syms x
f15 = @(x) exp((-x.^2)./2)
I = gaussQuad(f15, -s,s, 6,4)
A15 = 1/sqrt(2*pi)*I
```

I used the following two commands to obtain the desired results:

```
S = [1 2 2.5 3]
for i = 1:4, Q15DR(S(i)), end
```

and the probabilities I got are: 0.6827, 0.9545, 0.9876, 0.9973.

11-20 Here is the m-file I created

```
function Q20DR
% This m-file answers question 20 of Chapter 11 (p. 670) in Recktenwald
% It uses gaussquad to evaluate the integral x*exp(-x)

syms x;
f20 = @(x) x.*exp(-x);
I = eval(int(f20(x),0,5));
fprintf('panels error \n');
for k = [1 2 3 4 6 8]
    A = gaussQuad(f20, 0,5, k,2);
    Err = abs(I-A);
    fprintf('%d %f \n',k,Err);
end
```

and here is the output I got

panels	error
1	0.149789
2	0.020001
3	0.004669
4	0.001571
6	0.000324
8	0.000104

Since we use a two point Gaussian the error term is predicted to be $O(H^4)$. As explained in section 11.2 page 610 of Recktenwald, the truncation should be interpreted as the proportional decrease in the error as H decreases. For example if H is halved then the error should be $\frac{1}{2^4}$ smaller.

Using the Recktenwald function demoGauss, I obtained the following output:

```
Gauss-Legendre quadrature with 2 nodes
```

panels	H	I	error	alpha
1	5.00000	1.1093615949	1.50e-01	
2	2.50000	0.9795733101	2.00e-02	2.90
3	1.66667	0.9642415878	4.67e-03	3.59
4	1.25000	0.9611429211	1.57e-03	3.79
5	1.00000	0.9602344590	6.62e-04	3.87
6	0.83333	0.9598967252	3.24e-04	3.91
7	0.71429	0.9597491106	1.77e-04	3.94
8	0.62500	0.9596765991	1.04e-04	3.95

The term alpha is used to determine the exponent applied to the panel size for expressing the error. Letting H_1 and H_2, E_1 and E_2 represent the old and new H and E

values respectively, the formula used to compute alpha is

$$\alpha = \frac{\log(\frac{E_2}{E_1})}{\log(\frac{H_2}{H_1})}$$

As we see the value of alpha approaches the predicted value 4.

11-21 (NOTE: This question was not assigned but it is of interest.)

Repeating the preceeding problem using demoGauss with 4 and 8 nodes respectively I get the following output:

```
Gauss-Legendre quadrature with 4 nodes

panels     H            I              error        alpha
   1    5.00000    0.9601822632      6.10e-04
   2    2.50000    0.9595769057      4.59e-06        7.05
   3    1.66667    0.9595725249      2.07e-07        7.64
   4    1.25000    0.9595723398      2.18e-08        7.82
   5    1.00000    0.9595723218      3.76e-09        7.89
   6    0.83333    0.9595723189      8.86e-10        7.92
   7    0.71429    0.9595723183      2.60e-10        7.95
   8    0.62500    0.9595723181      8.99e-11        7.96
```

The alpha values go to 8 as expected.

```
Gauss-Legendre quadrature with 8 nodes

panels     H            I              error        alpha
   1    5.00000    0.9595723180      1.58e-11
   2    2.50000    0.9595723180     -2.22e-16       16.12
   3    1.66667    0.9595723180     -6.66e-16       -2.71
   4    1.25000    0.9595723180     -9.99e-16       -1.41
   5    1.00000    0.9595723180     -8.88e-16        0.53
   6    0.83333    0.9595723180     -8.88e-16       -0.00
   7    0.71429    0.9595723180     -7.77e-16        0.87
   8    0.62500    0.9595723180     -1.11e-15       -2.67
```

There is a problem computing alpha because the approximations are so close to the right answer that there is extremely little improvement when increasing the number of panels. The first value of alpha = 16.12 reflects the predicted theoretical value.

11-22 I wrote (by modifying an m-file that comes from Recktenwald) the following m-file.

3

```
function compintsinx2DR(a,b)
% compint_sinx2 Compare quadrature rules for integral of (sin(x))^2
%
% Synopsis: compint_sinx2
% compint_sinx2(a,b)
%
% Input: a,b = (optional) limits of integration. Default: a=0; b=pi
%
% Output: Values of integral obtained by trapezoid and simpsons rules
% for increasing number of panels
%
% NOTE: This function is modelled after compIntRules by Gerald W.
% Recktenwald.

if nargin < 2 , a = 0 ; b = 2*pi; end
xmin = a;  xmax = b;
fun = inline('sin(x).^2');              %  integrand, f(x)
ifun = inline('-1/2*cos(x)*sin(x)+1/2*x');      %  integral of f(x)
Iexact = ifun(xmax) - ifun(xmin);


np = [ 1 2 4 8 16];     %  number of panels

nft = np + 1;    %  number of function evaluations with Trapezoid rule
nfs = 2*np + 1;  %  ... with Simpson's rule
nfg = 2*np;    %  ... with Gauss-Legendre rule

% --- Compute truncation errors
for k=1:length(np);
  errt(k) = trapezoid(fun,xmin,xmax,np(k)) - Iexact;
  errs(k) = simpson(fun,xmin,xmax,np(k)) - Iexact;
  errg(k) = gaussquad(fun,xmin,xmax,np(k),2) - Iexact;
end


fprintf('\nTruncation error versus number of function evaluations\n\n');
fprintf('      Trapezoid            Simpson              %s\n',GLstr);
fprintf('    nf      error         nf      error         nf      error\n');
for k=1:length(np)
  fprintf('%6d  %11.2e  %6d  %11.2e  %6d  %11.2e\n',...
          nft(k),errt(k),nfs(k),errs(k),nfg(k),errg(k));
end
```

And then I ran it twice.

```
EDU>> compintsinx2DR(0,pi)


Truncation error versus number of function evaluations

         Trapezoid              Simpson              GL 2 node
      nf      error          nf      error         nf      error
       2    -1.57e+00         3    5.24e-01         2    -3.78e-01
       3     0.00e+00         5    0.00e+00         4     0.00e+00
       5     0.00e+00         9    0.00e+00         8     0.00e+00
       9     0.00e+00        17    2.22e-16        16     0.00e+00
      17     0.00e+00        33    2.22e-16        32     0.00e+00
EDU>> compintsinx2DR(0,2)


Truncation error versus number of function evaluations

         Trapezoid              Simpson              GL 2 node
      nf      error          nf      error         nf      error
       2    -3.62e-01         3    3.05e-02         2    -2.10e-02
       3    -6.77e-02         5    1.19e-03         4    -8.00e-04
       5    -1.60e-02         9    6.77e-05         8    -4.52e-05
       9    -3.96e-03        17    4.14e-06        16    -2.76e-06
      17    -9.86e-04        33    2.57e-07        32    -1.71e-07
```
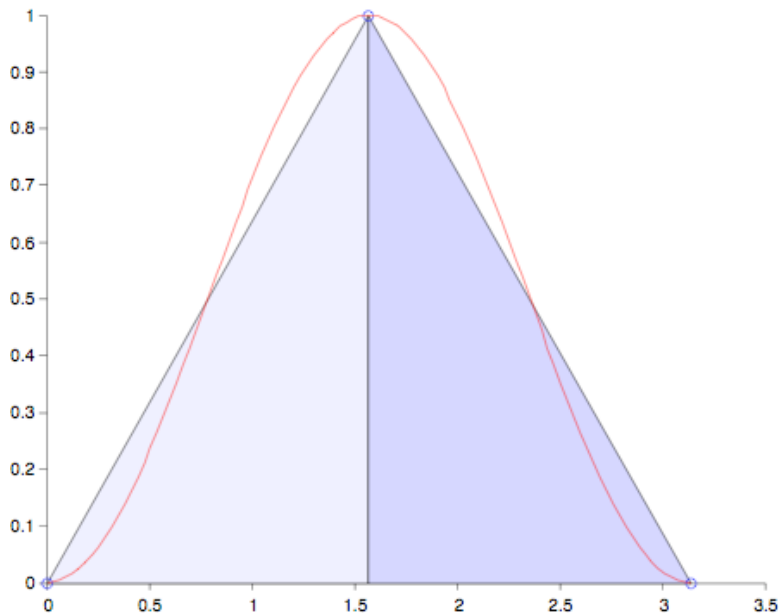
The symmetry of the function gives surprisingly good results when the limits of integration are $[0 \ldots 2\pi]$. Here is a picture that I generated using plottrapint that gives some idea of how symmetry cancels out the truncation error.

Recktenwald has more to say about this question. Taken from his online solutions manual.

$\boxed{\textbf{11.22}}$ Write an m-file function that evaluates $\int_0^{2\pi} \sin^2(x)\, dx$ using the composite trapezoid rule, composite Simpson's rule, and composite Gauss–Legendre quadrature with four nodes per panel. Place the calls to `trapezoid`, `simpson`, and `gaussQuad` inside a loop and repeat the calculations for `np = [2 4 8 16 32 64]`, where `np` is the number of panels. Record the number of function evaluations, `n`, for each method. Print the absolute error $|I - I_{\text{exact}}|$ for the three methods versus n. (See, for example, [13,§ 2.9] for help in explaining the results.)

Partial Solution: The computations are carried out with the `compint_sinx2` function. The prologue for `compint_sinx2` is

```
function compint_sinx2(a,b)
% compint_sinx2  Compare quadrature rules for integral of (sin(x))^2
%
% Synopsis:  compint_sinx2
%            compint_sinx2(a,b)
%
% Input:     a,b = (optional) limits of integration.  Default: a=0; b=pi
%
% Output:    Values of integral obtained by trapezoid and simpsons rules
%            for increasing number of panels
```

Running `compint_sinx2` with the default inputs gives:

```
>> compint_sinx2

        Integral of (sin(x))^2 from 0*pi to 1*pi
        Iexact =    1.570796327

    Trapezoid              Simpson              Gauss-Legendre
   n       error        n       error        n       error
   3     0.00000e+00     5     0.00000e+00     8     0.00000e+00
   5     0.00000e+00     9     0.00000e+00    16     0.00000e+00
   9     0.00000e+00    17     0.00000e+00    32     0.00000e+00
  17     0.00000e+00    33     0.00000e+00    64    -2.22045e-16
  33     0.00000e+00    65     0.00000e+00   128     0.00000e+00
  65     0.00000e+00   129     0.00000e+00   256     0.00000e+00
```

Note that `n` is the number of nodes at which the integrand is evaluated, not the number of panels used by the various composite rules. All methods give neglible errors regardless of the number of panels. The trapezoid rule is known to rapidly converge for a periodic integrand when the limits of the integral are points at which the integrand and its derivative assume periodic values. See Davis and Rabinowitz [13,§ 2.9] for details. $\sin^2 x$ is an extreme example of the special behavior for periodic integrands.

A slightly more interesting results is obtained if `np = [ 1 2 4 8 16 32]` and the Gauss–Legendre rule with two (instead of four) points per panel is used. Making these changes and running the modified `compint_sinx2` gives

```
>> compint_sinx2

    Integral of (sin(x))^2 from 0*pi to 1*pi
    Iexact =    1.570796327
```

Taken from Recktenwald's on line solutions manual.

	Trapezoid		Simpson		Gauss-Legendre
n	error	n	error	n	error
2	-1.57080e+00	3	5.23599e-01	2	-3.77963e-01
3	0.00000e+00	5	0.00000e+00	4	0.00000e+00
5	0.00000e+00	9	0.00000e+00	8	0.00000e+00
9	0.00000e+00	17	0.00000e+00	16	0.00000e+00
17	0.00000e+00	33	0.00000e+00	32	0.00000e+00
33	0.00000e+00	65	0.00000e+00	64	0.00000e+00

This result allows direct comparison with somewhat more comparable number of nodes in each row. The first row shows the error for applying the basic rule for each method.

If, for the integral in this Exercise, the limits of the integral are shifted to 0 and some non-rational multiple of π, the integration schemes behave as usual. Rerunning the modified version of `compint_sinx2` gives

```
>> compint_sinx2(0,5.12*pi)

    Integral of (sin(x))^2 from 0*pi to 5.12*pi
    Iexact =    7.871340417
```

	Trapezoid		Simpson		Gauss-Legendre
n	error	n	error	n	error
2	-6.78146e+00	3	2.83874e+00	2	-7.23535e+00
3	4.33692e-01	5	-6.75466e-01	4	2.67829e-01
5	-3.98177e-01	9	5.76797e-01	8	-4.38233e-01
9	3.33054e-01	17	-2.84133e-02	16	1.95645e-02
17	6.19534e-02	33	-1.10178e-03	32	7.40447e-04
33	1.46620e-02	65	-6.25700e-05	64	4.17971e-05

Now the truncation errors do not decrease so dramatically because the limits of the integral do not produce periodic values of the integrand and its derivatives.

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