On Convex Decompositions of Points

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Abstract. Given a planar point set in general position, S, we seek a partition of the points into convex cells, such that the union of the cells forms a simple polygon, P, and every point from S is on the boundary of P. Let $f(S)$ denote the minimum number of cells in such a partition of S. Let $F(n)$ be defined as the maximum value of $f(S)$ when S has n points. In this paper we show that $\lceil (n-1)/4 \rceil \leq F(n) \leq \lfloor (3n-2)/5 \rfloor$.

1 Introduction

Partitions of point sets into convex subsets is a ubiquitous problem in discrete geometry. The domain is a finite set of points in the plane, which we will usually denote by S. The points are assumed to be in general position, that is, no three points on a line. A subset of S that are the vertices of a convex k-*gon* is a *convex subset*. A convex subset of S with no points of S in its open interior is called an *empty convex subset*. The landmark paper of Erdős and Szekeres [3] asks for the value of the smallest integer $A(k)$ such that any set of $A(k)$ points contains a convex subset of size k. Subsequently a similar question is asked by Erdős in [2] for the value of the smallest integer $B(k)$ such that any set of $B(k)$ points contains an empty convex subset of size k. Values for $B(k)$ are known for all values of k except $k = 6$. In [2] it is shown that $B(3) = 4$, and $B(4) = 5$. In [4] it is shown that $B(5) = 10$. Horton [5] gives a construction showing that $B(7)$ is not finite, that is, there are arbitrarily many points with no empty convex 7-gons. Horton's construction for 16 points is shown in Figure 1. The value of $B(6)$ is not known, and this remains a tantalizing long outstanding open problem. Some experimental results showing that $B(6) > 20$ as well as an algorithm for computing maximum empty convex subsets were first shown in [1]. A 26 point set with no empty hexagons is given in [6]. Some combinatorial results on partitions of points sets in two and three dimensions, are presented in [7] and [8].

In this paper we consider the following variation on the convex partition theme. Given a set of points S we want to partition S into empty convex subsets such that the union of the subsets form a single simple polygon P, and every point in S is on the boundary of P. Here, we call such an empty convex subset of k-gon in P a k-*cell*. Given S, let $f(S)$ represent the minimum number of cells obtained in such a partition of S. Let $F(n)$ denote the maximum value of $f(S)$ over all sets S with n points.

For example, a 16 point set S in Figure 1 gives $f(S) = 4$. For any simple polygon with order n, since we can always triangulate its interior, the trivial upper bound of $F(n)$ is $n - 2$.

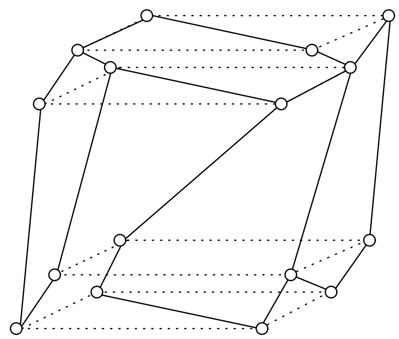

Fig. 1. Horton's construction with 16 points, partitioned into four convex cells.

In the next section we prove the following theorem.

Theorem 1. $\left\lceil \frac{n-1}{4} \right\rceil \leq F(n) \leq \left\lfloor \frac{3n-2}{5} \right\rfloor$

2 Upper and lower bounds

As was shown in [5] there exists sets $n = 2^k$ with no empty convex hexagons giving the lower bound $n/4$. We obtain here the lower bound for any integer n.

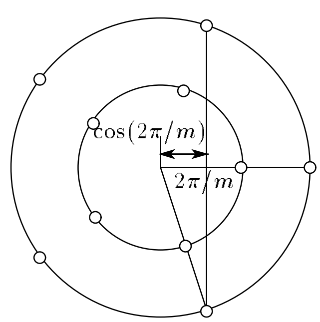

Fig. 2. An example to illustrate the lower bound

Lemma 1. $F(n) \geq \left\lceil \frac{n-1}{4} \right\rceil$.

Proof. We first construct a set of $n = 2m$ points, such that we have m points equally space on a circle of radius 1, and the other m points on a circle of radius $\cos(2\pi/m) + \epsilon$, where ϵ is a small positive value. If necessary we perturb the points slightly so that no 3 points are on the same line. See Figure 2. If n is odd, place an additional point close to the center. This ensures that there are no empty triangles made up of points on the outer circle. Thus no cell uses more than two points from the outer circle, achieving the desired bound. □

For the upper bound we present an iterative construction. Let $CH(S)$ denote the anti-clockwise circular sequence of points when traversing the vertices of the convex hull of S. The input to the construction are the points of S and two consecutive points on $CH(S)$ where the line segment between the two points is called a *starting edge*. We induce a permutation of the points $\Pi(S) = (1, 2, 3, \ldots n)$. Thus let points 1 and 2 be a fixed pair of consecutive points on $CH(S)$. The rest of the sequence is obtained by sweeping a ray anchored at 1, and passing through 2, 3, .., n, which is implicitly the other neighbour of 1 on $CH(S)$. See Figure 3. We assume that $n3$. It is easy to see that $F(3) = 1$ and $F(4) \leq 2$. Using the fact that every 5 points has an empty convex 5-gon, that is, B(4) =5, we deduce that $F(5) \leq 2$ and $F(6) \leq 3$. For values of $n \geq 7$ we use an iterative approach.

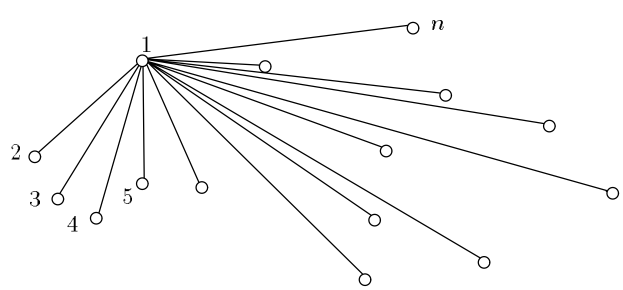

Fig. 3. The points in S are ranked according their order in an angular sweep of a ray anchored at 1.

After each iteration there are two basic outcomes.

In the first instance we look for a point w so that $123w$ is a 4-cell. Then the points 3 and w are neighbours in $CH(S \setminus \{1, 2\})$, and the edge $3w$ is the starting edge for the next iteration. Observe that 3 is always in $CH(S \setminus \{1, 2\})$. If $\triangle 13n$ is empty, then set w to n, otherwise set w to the neighbour of 3 on $CH(S \setminus \{1, 2\})$, contained in $\triangle 13n$. If $123w$ is convex, then it is also empty and we are done. See Figure 4.

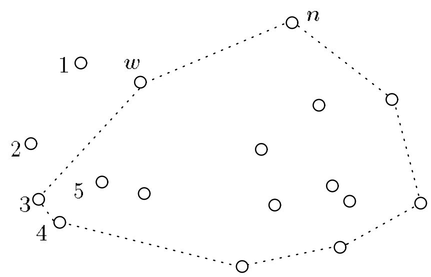

Fig. 4. The case where $123w$ is convex

Suppose $123w$ is not convex. The second possibility operates on a 7 point subset. Then we will find either 6 or 7 point subset S' such that S' can be decomposed into 2 or 3 cells, respectively, so that one of the cells is $\triangle 123$, and another cell uses the edge uv so that u and v are consecutive on $CH(S \setminus S' \cup \{u, v\})$. Thus we can use uv as the starting edge for the next iteration. We will detail the decomposition process on this 7 point subset. Observe that the maximum number of cells produce by this iterative process is $\lfloor 3n - 2/5 \rfloor$.

Consider the lines passing through the points 1,3 and 2,3. This partitions the plane into 4 regions plus the lines themselves. An important consequence of the fact that there is no fourth point that makes an empty convex 4-gon with 1,2,3 is that one of these 4 regions contains all of $S \setminus \{1, 2, 3\}$. See Figure 5.

Let $S_0 = \{4, 5, 6\}$ and $S_1 = S \setminus \{1, 2, 3, 4, 5, 6\}$. Consider the inner common tangent line L of $CH(S_0)$ and $CH(S_1)$ such that one half plane bounded by L contains $\{1\}$ and S_0. Let p be a point in L and S_0 and q be a point in L and S_1. Let the two points in $S_0 \setminus \{p\}$ be called s and t. See also Figure 5. There are two primary cases to consider.

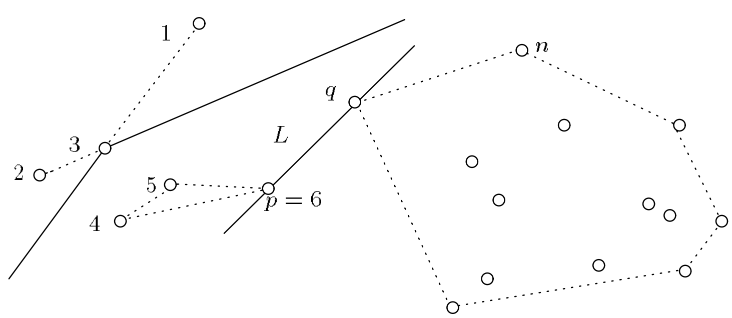

Fig. 5. We find a pair of points p and q that are in $CH(S \setminus S' \cup \{p, q\})$.

Case A) The line through s and t does not intersect the line segment between p and q. Then $pqst$ is a 4-cell. For convenience we rename the points q_0, q_1, q_2, q_3 as they appear in anti-clockwise order on the boundary of the 4-cell with $q = q_0$, where $q_3 = p$. See Figure 6. If a line through q_0, q_1 does not intersect the edge 13, then we obtain a 4-cell $q_0 13 q_1$ to add to $pqst$ and $\triangle 123$.The starting edge for the next iteration is $q_0 q_3$. Otherwise, let i be the largest integer such that the directed ray anchored at q_{i-1} and passing through q_i, intersects the edge 13. If $i \neq 3$ we set $j = i + 1$. If the line through q_i, q_j also intersects with the edge 13, then we use three cells $q_i 32 q_j, q_0 q_1 q_2 q_3$ and $\triangle 123$, and $q_0 q_3$ is the starting edge for the next iteration. If q_i, q_j does not intersect with the edge 13 we use $q_i 13 q_j, q_0 q_1 q_2 q_3$ and $\triangle 123$ with the starting edge $q_0 q_3$. If $i = 3$, the line through q_0, q_3 necessarily intersects the edge 13. In this case we obtain two cells, a 5-cell $q_1 q_2 q_3 32$ and $\triangle 123$, and use $2q_1$ as the starting edge for the next iteration.

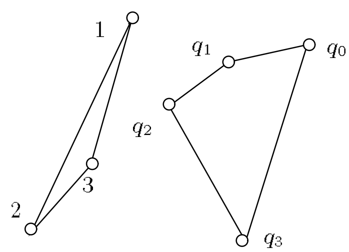

Fig. 6.

Case B) The line through s and t intersects the line segment between p and q. Let t be the point closest to the segment pq. Denote the anti-clockwise angle of (x, y, z) for the points x, y, z by $\angle(x, y, z)$. If $\angle(3, s, t)$ is convex, then we obtain three cells $3stq1, ptq$ and $\triangle 123$ with the starting edge pq for the next iteration. Otherwise, we examine the pentagon $2pts3$, and if it is convex we use it with $\triangle 123$ and tpq. The starting edge for the next iteration is pq. However, $2pts3$ may not be convex, it may not even be simple. Thus we need to examine a few more sub-cases.

We enumerate the possible cases according to which of p, s, t is equal to 6.

1. $p = 6$: This implies that $2pts3$ is convex so nothing more needs to be done.
2. $s = 6$: Let $r \in S_1$ such that the angle $\angle(r, s, 1)$ is maximized. Since the line through s and t intersects the line segment between p and q we deduce that $p < t$. See Figure 7(b). We obtain cells rts, and $\triangle 123$, and we use sr as the starting edge for the next iteration. If $\angle(t, p, 3)$ is convex then we also use the 5-cell $13pts$. On the other hand if $\angle(3, p, t)$ is convex we use the 5-cell $2rtp3$.

3. $t = 6$: Let $r, r' \in S_1$ such that $\angle(r, t, 1)$ and $\angle(r', t, 1)$ are maximum and minimum, respectively. We consider the situation according to the position of p. Note that $p \notin \triangle 1st$. See Figure 7(a).
   (a) $p \in \triangle 13s$. We obtain the two cells $\triangle 123$ and $32tsp$, and use the starting edge $2t$ for the next iteration.
   (b) $p \in \triangle 23s$. Two of the three desired cells are $\triangle 123$ and $13ps$. At least one of $tspr$ or $1str'$, must be convex. If the 4-gon $tspr$ is convex, choose this as the third cell and tr as the starting edge for the next iteration. Otherwise, choose $1str'$ as the third cell and tr' for the starting edge.
   (c) $p \in \triangle 2st$. We obtain three cells $123, 32ps$ and $2rtp$, and the edge tr is the starting edge for the next iteration.

Thus this construction proves the following lemma.

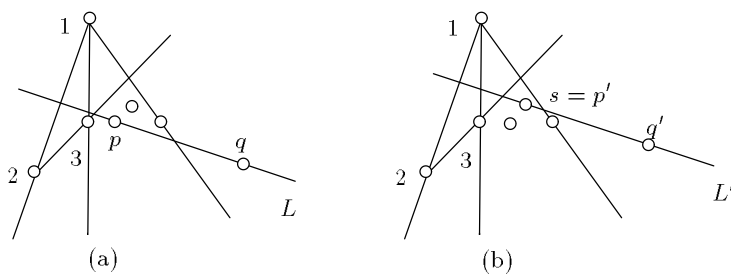

Fig. 7.

Lemma 2. $F(n) \leq \left\lfloor \frac{3n-2}{5} \right\rfloor$.

Combining lemmas 1 and 2 proves our theorem.

3 Discussion

We give upper and lower bounds for a variation of convex partitions. There is still a substantial spread between the upper and lower bound. From our point of view a more complicated algorithm may be able to improve the upper bound somewhat, but we conjecture that the correct bound is $n/2$. The lower bound although easy to prove seems to be more difficult to improve upon, as there seems to be a connection with settling the issue of whether $B(6)$ is finite or not. Observe that, $F(n) < (n-2)/3$ for all but finitely many n, implies the finiteness of $B(6)$.

4 Acknowledgements

We would like to thank the referee for the careful reading of the manuscript and the many helpful comments.

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