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On the visibility graph of convex translates

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Abstract

We show that the visibility graph of a set of non-intersecting translates of the same compact convex object in $\mathbb{R}^2$ always contains a Hamiltonian path. Furthermore, we show that every other edge in the Hamiltonian path can be used to obtain a perfect matching that is realized by a set of non-intersecting lines of sight. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let S denote a set of non-intersecting compact convex objects in $\mathbb{R}^2$. We say that two objects a and b from the set S *see each other* if there exists a straight line segment l with one point in a and one point in b such that l lies in the complement of $S - \{a, b\}$. We call such a line segment a *line of sight*. The *visibility graph* of S, denoted by $\mathrm{Vis}(S)$, associates a vertex to each object of S, and an edge between two vertices if and only if the associated objects see each other.

The combinatorial structure of the visibility graph for sets in $\mathbb{R}^2$ has been studied extensively. Some results on the combinatorial structure of the visibility graph of line segments can be found in [10,4,8,7]. For a survey of results pertaining to visibility problems see the Handbook of Discrete and Computational Geometry [6].

In [7] the notion of a set of equal width objects is introduced. Let s denote a compact object in $\mathbb{R}^2$. The f-*width* of s is the perpendicular distance between two distinct parallel lines of support of s with direction f. Two compact objects in $\mathbb{R}^2$ have *equal width* if pairwise their f-widths are equal for every value f. Hosono and Matsuda [7] conjecture that the visibility graph of any even cardinality set of disjoint

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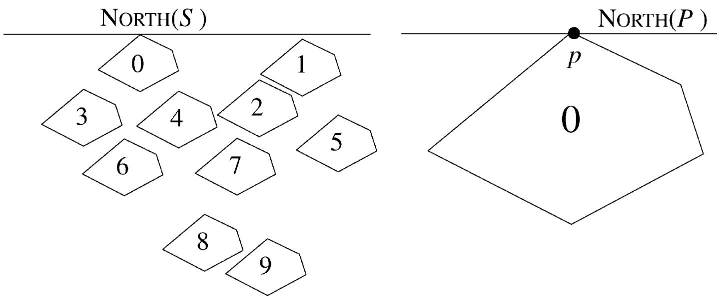

Fig. 1. Consider S a set of 10 polygons, and let P denote the set of points contained in polygon 0. Then north$(S) = 0$ and north$(P) = p$.

convex equal width objects has a perfect matching, realized by non-crossing lines of sight.

In this paper we strengthen the notion of equal width to guarantee that the visibility graph of a set of convex objects with these properties contain a Hamiltonian path, and by extension a perfect matching when the cardinality of the set is even. Our class of objects contains any set of translates of the same compact convex object in $\mathbb{R}^2$. Our main results are found in Theorems 3.9 and 4.2.

2. Preliminaries

This section will provide some of the basic definitions we use and introduce some notation.

Let $l : ax + by = c$ denote an infinite line, and let l^+, and l^-, denote half-planes bounded by l and respectively satisfying $ax+by \geqslant c$ and $ax+by < c$. Let Γ denote an arbitrary compact subset of $\mathbb{R}^2$. We say that l *supports* Γ if $l \cap \Gamma \neq \emptyset$ and, $l^+ \cap \Gamma = \Gamma$. Note, the direction of a line of support l is implicit, as we always use $l^+(\Gamma)$ to denote the closed half-plane that contains Γ and $l^-(\Gamma)$ to denote the open half-plane that is disjoint from Γ. If $\gamma \in \Gamma$ is contained in l then we say that γ is *extreme*.

For a compact set Γ we define North(Γ) as a horizontal support line above Γ. We use north(Γ) to denote the leftmost element, of Γ contained in North(Γ). For example, let Γ denote a convex polygon in $\mathbb{R}^2$, Then North(Γ), and north(Γ) respectively denote a horizontal support line above Γ, and the leftmost point on the boundary of Γ contained in North(Γ). We use the same notation when Γ is used to denote a set of compact convex subsets of $\mathbb{R}^2$. Then north(Γ) denotes an element $\gamma \in \Gamma$ that is incident to a horizontal support line above $\bigcup_{\gamma \in \Gamma} \gamma$, which is called North$(\Gamma)$. See Fig. 1.

By analogy we define support lines, and extreme elements of a set Γ with respect to the other compass points, using South(Γ), south(Γ), East(Γ), east(Γ), West(Γ), and west(Γ).

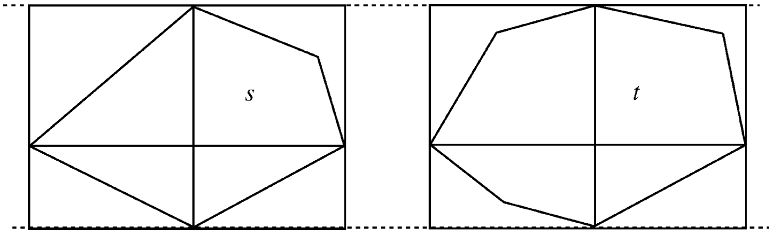

Fig. 2. Objects s and t are of axis equal width. That is, $d(\text{north}(t), \text{centre}(t)) = d(\text{north}(s), \text{centre}(s))$, $d(\text{south}(t), \text{centre}(t)) = d(\text{south}(s), \text{centre}(s))$, $d(\text{east}(t), \text{centre}(t)) = d(\text{east}(s), \text{centre}(s))$, and $d(\text{west}(t), \text{centre}(t)) = d(\text{west}(s), \text{centre}(s))$. The objects s and t are also axis aligned.

Let s be used to denote a compact convex subset of $\mathbb{R}^2$. We say that s is *axis oriented* if the line passing through north(s) and south(s) is vertical, and the line passing through east(s) and west(s) is horizontal, that is, these lines are parallel to a rectilinear bounding box of s.

We define the *centre* of s, denoted by centre(s), as the point of intersection of the line through north(s) and south(s) and the line through east(s) and west(s). Let $d(p,q)$ denote the Euclidean distance between two points p and q. We say that two axis oriented convex objects s and t are of *axis equal width*, if the distance from the centre of s to any of its compass points is equal to the corresponding distance in t. See Fig. 2.

Let S denote a set of at least two compact convex objects in $\mathbb{R}^2$. For $s,t \in S$ we say that s and t are *axis aligned*, if both $\textsc{North}(s)=\textsc{North}(t)$ and $\textsc{South}(s)=\textsc{South}(t)$, or if both $\textsc{East}(s)=\textsc{East}(t)$ and $\textsc{West}(s)=\textsc{West}(t)$.

We define a set, S, of disjoint compact convex subsets of R^2, a *non-aligned axis-oriented* set, if every $s \in S$ is axis-oriented, and every pair $s,t \in S$ are of axis equal width, and not axis aligned.

We show that if S is a non-aligned axis-oriented set then Vis(S) contains a Hamiltonian path. Barring axis alignment is not essential, however, it will simplify the discussion if we adhere to sets that are not axis aligned. In Section 4 we show that our results can be applied to any set of translates of the same compact convex object in $\mathbb{R}^2$.

We now set down some necessary graph theory terminology. A standard reference is [3]. We use $G=(V,E)$ to denote a graph with vertex set V and edge set E with no loops or multiple edges. We denote edges by two element subsets of the vertex set. A graph $G=(V,E)$ is *planar* if the vertices V can be positioned on a plane, so that all edges E can be realized by non-crossing straight line segments and no vertex is interior to any edge. We call this a *drawing* of G. A drawing of a planar graph partitions a plane into disjoint regions, or *faces*. There is exactly one face that is unbounded, and this is called the *outer* face. A drawing of a planar graph is a *triangulation* if all of its faces, except possibly the outer face, are triangles. (Note: the definition that we use for triangulation is slightly non-standard, as we include in our definition graphs that are not two-connected, that is the outer face may not have a two-connected boundary.) A triangulation with an outer face that is a triangle is called *maximal planar*. We say that $\{u,v,w\} \subseteq V$ is a *3-clique* if $\{\{u,v\},\{v,w\},\{u,w\}\} \subseteq E$. A 3-clique in a triangulation is called a *a separating triangle* if it contains vertices both in its interior and exterior.

In [12] Whitney has shown that a maximal planar graph with no separating triangles is Hamiltonian. A graph G is an *NST triangulation* if it has a drawing that is a triangulation, and it has no separating triangles. We use *QNST triangulation* to denote an NST triangulation with an outer face that is a quadrangle. In [5] Dillencourt extends Whitney's results in several ways. We use Dillencourt's result that every QNST triangulation is Hamiltonian and there is always a Hamiltonian cycle in it that passes consecutively through three adjacent edges of its outer face.

In Section 3 we show that given a non-aligned axis-oriented set, S, we can always find a subgraph, M, of Vis(S) that is an NST triangulation. Furthermore, if M has an outer face of more than four vertices, then we can augment M with four additional vertices, to create a QNST triangulation. We then use the results in [5,12] to show that the visibility graph of a non-aligned axis-oriented set admits a Hamiltonian path. We also show that there is a perfect matching that uses non-crossing lines of sight. Let s be a compact convex subset of the plane. We say that S is a set of *convex translates* (of s) if every element of S is a translation of the model s. In Section 4 we show that our results apply for any set of convex translates.

3. An NST subgraph of Vis(*S*)

We assume throughout that S denotes a non-aligned axis-oriented set. Consider a graph M obtained from S. For each object in S we associate a corresponding vertex in M. Our goal is to construct M, so that M is both a subgraph of Vis(S) and an NST triangulation. We first give an informal description of the graph M, and follow it with an algorithm that precisely shows how M is constructed.

For expository purposes we partition the edges of M into three classes. We use s when referring to the vertex associated to the object $s \in S$ and assume that the meaning of the symbol will be clear from the context. We use st to denote an edge between the vertices s and t, in M.

An edge st in class I connects two vertices if there exists a horizontal line of sight between objects s and t.

An edge st in class II has the property that $\textsc{South}^{-}(s) \cap \textsc{North}^{-}(t)$ is not the empty set and is contained in the complement of S. We call the area defined by $\textsc{South}^{-}(s) \cap \textsc{North}^{-}(t)$ a *clear corridor*.

The remaining edges of M, those of class III, are added by an iterative process. We say that s is maximal in the east direction, or eastmost, if there exists a horizontal line l such that $l \cap s$ contains the rightmost point in a non-empty intersection of l and S.

Let r, s, and t denote three distinct eastmost objects of S so that s is above r and r is above t, r is to the left of both s and t, and both of sr and rt are edges in M. In

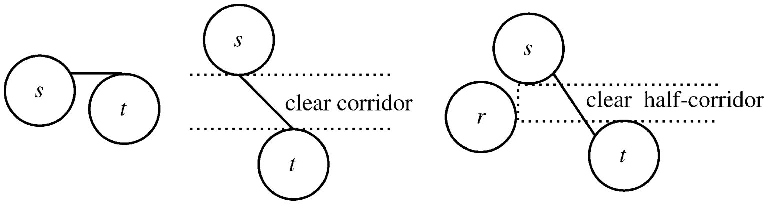

Fig. 3. Edges of class I, II, and III are illustrated from left to right.

such a situation we add the class III edge st to M. We show in Lemma 3.2 that the region bounded by $\text{East}^{-}(r) \cap \text{South}^{-}(s) \cap \text{North}^{-}(t)$ is non-empty and is contained in the complement of S, so we call it a *clear half-corridor*. This iterative process of adding edges will be made precise by the algorithm used in procedure Make-III described below. Similarly we can define edges for westmost triples. See Fig. 3 for an illustrative example.

We give a pseudo code algorithm that computes the graph M using standard terminology. The algorithm uses the plane sweep technique common to many algorithms in computational geometry [2,9]. The algorithm sweeps a horizontal line from the top to the bottom of the plane. A priority queue, Q, using points from $\{\text{north}(s), \text{south}(s) \mid s \in S\}$ sorted by decreasing y-coordinate determines the discrete events that signal some action. (Note: Whenever north(t) and south(s) have the same y-coordinate we queue north(t) before south(s).) A search structure L maintains in x-coordinate order all objects currently intersected by the sweep line. We initialize L with dummy leftmost and rightmost entries, which we call α and ω. The algorithm also makes use of a pair of stacks called EastStack and WestStack that keep track of eastmost and westmost objects used in the construction of class III edges. A procedure Make-III is used to handle the creation of class III edges.

```
Make-M(Q,L)
loop
q ← Dequeue(Q)
case q of
q = north(s):
   Insert s into L
   {Let the left and right neighbours of s in L be respectively de-
   noted as s⁻ and s⁺.}
   if(s⁻ = α then Make-III(WestStack,s) else Add class I edge ss⁻ to M
   if(s⁺ = ω then Make-III(EastStack,s) else Add class I edge ss⁺ to M
q = south(s):
   Delete s from L.
   if (s = south(S) return
{Let the former left and right neighbours of s in L be respectively
denoted as s⁻ and s⁺.}
```

if ($s^- = \alpha$ **and** $s^+ = \omega$ Add the class II edge su to M, where north(u) is the next object in Q.
if $s^- = \alpha$ **and** $s^+ \neq \omega$ Make-III(WestStack,s^+)
if $s^- \neq \alpha$ **and** $s^+ = \omega$ Make-III(EastStack,s^-)
if $s^- \neq \alpha$ **and** $s^+ \neq \omega$ Add class I edge s^-s^+ to M
endloop

We now give the pseudo code for procedure Make-III used to create the class III edges. We use standard stack operation Push and Pop. We augment the standard stack operations with an operation Size(Stack) that returns the number of elements on the stack. We use the notation Stack(top) to obtain the value of the top element of the stack, without popping it from the stack. Similarly Stack(top $-$ 1) is used to access the value of the next to top element of the stack. The procedure Make-III loops as long as class III edges can be added to M.

Make-III (Stack,t)
1. **loop**
2. **if** (Size(Stack) $<$ 2 Push (Stack,t) **return**
3. $r \leftarrow$ Stack(top) $s \leftarrow$ Stack(top $-$ 1)
4. **if** (Stack = EastStack) **then** $l_r = \textsc{East}(r)$ **else** $l_r = \textsc{West}(r)$
5. **if** $l_r^- \cap s \neq \emptyset$ **and** $l_r^- \cap t \neq \emptyset$) **then**
6. Add the edge st to M
7. Pop(Stack)
else
8. Push(Stack,t) **return**
end loop

We will analyse procedure Make-III assuming operations on the stack EastStack. Clearly symmetric arguments apply for the WestStack. Let $e_1, e_2, \ldots, e_m$ denote the sequence of eastmost objects in S ordered from top to bottom. Observe that e_ie_{i+1}, for $i = 1, \ldots, m-1$ is always a class I or class II edge in M.

Lemma 3.1. *Upon entering procedure Make-III, if* Size(EastStack)$\geqslant$2 *then* {Stack (top), Stack(top $-$ 1)} is an edge in M.

Proof. Observe that if Make-III is called with $t = e_i$ then upon entering Make-III the top of a non-empty EastStack contains e_{i-1}. This follows from the fact that t is pushed onto the stack in all cases prior to returning from Make-III. Therefore, if no objects are popped from the stack our lemma holds. In the case that we do pop elements from the stack, see line 7 of Make-III, we add the edge st. Thus when we finally exit Make-III and push t onto the stack, we have added the edge st to M, where s is the value Stack(top$-$1) the next time we enter procedure Make-III. Since all stack operations on EastStack occur in Make-III we conclude that the lemma holds. □

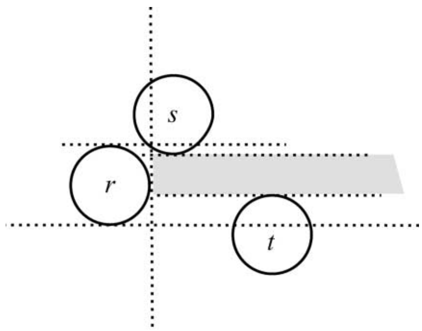

Fig. 4. The region $\text{East}^{-}(r) \cap \text{South}^{+}(r) \cap \text{North}^{+}(r) \cap \text{South}^{-}(s) \cap \text{North}^{-}(t)$.

Consider the situation in procedure Make-III where we add the edge st to M. We call the region bounded by $\text{East}^{-}(r) \cap \text{South}^{-}(s) \cap \text{North}^{-}(t)$ a *clear half-corridor*. We prove in Lemma 3.2 that this region is non-empty and is in fact contained in the complement of S.

Lemma 3.2. *Consider r, s and t as above such that a class III edge is added in Make-III using the* EastStack *stack. Then* $\text{East}^{-}(r) \cap \text{South}^{-}(s) \cap \text{North}^{-}(t)$ *is not the empty set and is contained in the complement of* S.

Proof. If $s = e_{i-2}$, $r = e_{i-1}$, and $t = e_i$, i.e., is s, r and t are consecutive eastmost objects, then the lemma holds because the regions bounded by $\text{East}^{-}(r) \cap \text{North}^{-}(r) \cap \text{South}^{-}(s)$ and $\text{East}^{-}(r) \cap \text{South}^{-}(r) \cap \text{North}^{-}(t)$ are either empty or are in the complement of S, and the region $\text{East}^{-}(r) \cap \text{South}^{+}(r) \cap \text{North}^{+}(r) \cap \text{South}^{-}(s) \cap \text{North}^{-}(t)$ is a non-empty subset of the plane and is in the complement of S. See Fig. 4. On the other hand, suppose we have $s = e_{i-k_2}$, $r = e_{i-k_1}$, where $k_2 > k_1 \geqslant 1$. Assume inductively that for all values of k such that $1 \leqslant k < k_2$ our hypothesis holds. Therefore, whether the edges rs and rt are of class I, II, or III we are guaranteed that the half-corridors $\text{East}^{-}(r) \cap \text{North}^{-}(r) \cap \text{South}^{-}(s)$ and $\text{East}^{-}(r) \cap \text{South}^{-}(r) \cap \text{North}^{-}(t)$ are either empty or are in the complement of S. As above $\text{East}^{-}(r) \cap \text{South}^{+}(r) \cap \text{North}^{+}(r) \cap \text{South}^{-}(s) \cap \text{North}^{-}(t)$ is non-empty and in the complement of S. Therefore the lemma holds. □

We now show that the edges we added to M are actually realized by lines of sight between objects of S.

Lemma 3.3. *M is a subgraph of* Vis(S).

Proof. The edges added to M in Make-M are realized by lines of sight in the obvious way. For class I edges, in lines 6 and 7 of Make-M an edge ss^{-} or an edge ss^{+} is realized by a horizontal line segment emanating from the point north(s), and in line 14 we use a horizontal line of sight between s^{-} and s^{+} that is just below s. For those

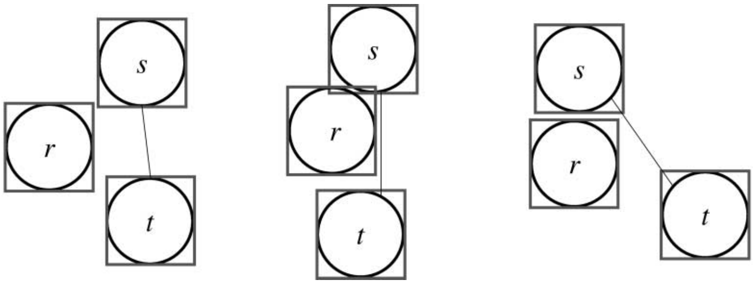

Fig. 5. The cases for realizing a class III visibility edge.

edges of class II of the form *su*, added in line 11 of Make-M, we use the line of sight with endpoints south(s) and north(u).

For the edges added in the procedure Make-III a more subtle argument is needed. To simplify the discussion let us assume that we are adding an edge *st* where r, s and t are on the EastStack.

Case 1: south(s) *sees* north(t). These points can be used as endpoints of a line of sight between s and t. Note: If the line of sight between south(s) and north(t) is blocked, then it must be blocked by r.

Case 2: south(s) *does not see* north(t) *and* s *and* t *are intersected by a common vertical line.* If s and t are intersected by a common vertical line, then we can always find a vertical intersecting line that misses r. Thus let h denote such a vertical intersecting line. We claim that a subset of h, with one endpoint p on the boundary of s and one endpoint q on the boundary of t is a line of sight. If there is any occlusion that blocks this line of sight, then by the clear half-corridor property, that occlusion must occur either above SOUTH(s), or below NORTH(t). We consider the possibility that an occlusion occurs above SOUTH(s) and show that this leads to a contradiction. A similar symmetric argument can be made against the case for an occlusion below NORTH(t).

Observe that the point p occurs on the boundary of s, clockwise between east(s) and west(s), and if there is any occluding object then it must intersect the region bounded by the boundary of s, SOUTH(s), EAST(s), and WEST(s), as illustrated in Fig. 5. Suppose there is an object $x \in S$ somewhere in this region. By the clear half corridor property south(x) is above south(s). However, since the objects are axis-oriented we have that east(x) and west(x), must both be above east(s) and west(s). So if x is in the region, then x also intersects s. However, our initial assumption is that our objects are disjoint. Therefore we have established the desired contradiction, and conclude that we have realized a valid line of sight between s and t.

Case 3: south(s) *does not see* north(t) *and* s, *and* t *are not intersected by a common vertical line.* If t is to the right of s then let h denote a line passing through the point of intersection of SOUTH(s) and EAST(s) and the point of intersection of NORTH(t) and WEST(t). A subset of h with endpoints on s and t is a line of sight between s and t.

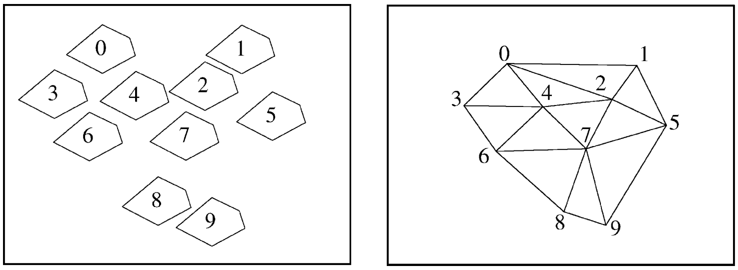

Fig. 6. A non-aligned axis-oriented set, with a drawing of its associated graph.

See Fig. 5. This claim is justified by an argument similar to the one given for case 2. If s is to the right of t then let h denote a line passing through the point of intersection of South(s) and West(s) and the point of intersection of North(t) and East(t), and we argue as above.

In conclusion we have shown that all edges in M can be realized by lines of sight, so M is a subgraph of Vis(S).

Once the actual lines of sight have been established it is a routine matter to obtain a planar drawing of M.

Lemma 3.4. *M is a planar graph.*

Proof. We prove that M is planar by giving a planar drawing of it. Each vertex of M is realized by its centre. Given the explicit lines of sight used to construct M, it is a routine matter to see that using the centres of each object as the vertex in a straight line drawing results in a planar graph. For an illustrative example see Fig. 6. We sketch some of the details. A class II edge requires a clear corridor, so clearly no class II edge can participate in a crossing. Suppose we have two class I edges that cross, and we call them e and f, such that the horizontal line of sight corresponding to e is above the horizontal line of sight corresponding to f. We label from right to left the objects incident to e and f e_1, e_2 and f_1, f_2. This implies that the f line of sight passes below at least one of the objects incident to e. Let us assume that the line passes below e_2, then the centres of the f_1 and f_2 are both below the centre of the e_1. So the centre to centre edges do not cross. Finally consider the possibility of a class III edge participating in a crossing. However, our argument regarding the clear half corridor property precludes any class III edges crossing. □

At the conclusion of the algorithm Make-M the stacks EastStack and WestStack are guaranteed to be non-empty. The objects north(S) and south(S) both appear on both stacks. All those objects on the boundary of the outer face of the graph M also remain on the stacks, and some (the ones that are cuts points) are on both stacks. Let D denote the number of objects that are found on both stacks at the conclusion of algorithm Make-M. Thus let $m = \text{Size(WestStack)} + \text{Size(EastStack)} - D$. Consecutive elements on EastStack, and WestStack represent edges on the outer face of M. In Lemma 3.1 we showed that consecutive elements on EastStack, and WestStack are connected by edges in M. Thus m represents the number of vertices on the outer face of M. Let n denote the cardinality of S. Recall that Euler's relation relates the cardinality of the sets of vertices, V, edges, E, and faces, F, in a planar graph, by the formula: $|V| - |E| + |F| = 2$.

Lemma 3.5. *M is a triangulation with an m sided outer face.*

Proof. Every time we encounter north(s) for an object in s we obtain currency for two potential edges, that is, we either add one of two edges and push one of two objects onto a stack. See lines 6–7 of Make-M. Stacked objects will either appear in our count as an edge for the case where the object eventually is removed from the stack or as a vertex of the outer face if the object remains on the stack at the end of the construction. This gives us a total of $2n - m - 2$ edges that are added when processing the north points.

For each point south(s) we obtain currency for one edge, that is, we either add the edge su, line 11 of Make-M, push an object onto a stack, line 12–13 of Make-M, or add the edge $s^- s^+$, line 14. This applies for points south(s), for all $s \in S$ except for south(south(S)). Therefore, we can add an additional $n - 1$ edges to our count.

The total number of edges in M is $2n + (n-1) - (m+2) = 3n - 3 - m$. We know that M is planar with an external face with m vertices, and thus using Euler's relation and our edge count we conclude that all faces of M except the outer face is a triangle. Therefore, M is a planar triangulation with an m sided outer face. $\square$

If the number of vertices on the outer face of M is three or four, and we show that M is an NST triangulation then we can use the results in [5] directly. Otherwise we augment the graph M with four additional vertices to obtain the desired combinatorial structure. Therefore, we first show that M is an NST triangulation.

Lemma 3.6. *M is an NST triangulation.*

Proof. Let $\{a,b,c\}$ be a 3-clique in M. Without loss of generality we assume that a is above b and b is above c. Let $\triangle(abc)$ denote the triangle formed by centre(a), centre(b), and centre(c). If $\{a,b,c\}$ is a separating triangle then there exists an object d such that $d \notin \{a,b,c\}$ and centre(d) is inside $\triangle(abc)$, and $\{a,b,c\}$ are not the vertices of the outer face of M.

Our analysis depends of the class of the edge ac.

Case 1: The edge ac is of class I. Since a sees c by a class I edge this implies that North(c) is above South(a). Since b is between a and c in the vertical ordering we do not have a clear corridor, so neither ab nor bc can be in class II and we similarly cannot have a clear half corridor for class III edges. Therefore both ab and bc must be in class I. If a sees c by a class I edge and a is above b and b is above c, then both a and c must be to the same side of b. But this implies that b cannot see both a and c with class I edges if centre(d) is in the interior of $\triangle(abc)$. Therefore, $\{a,b,c\}$ is not a separating triangle when ac is a class I edge.

Case 2: The edge ac is of class II. Since a is above b and b is above c, then ac cannot be a class II edge, because the region $\textsc{South}^-(a) \cap \textsc{North}^-(c)$ intersects with b and thus is not contained in the complement of S. Thus $\{a,b,c\}$ is not a separating triangle when ac is a class II edge.

Case 3: The edge ac is of class III. It simplifies the discussion, without losing generality, by assuming that ac is derived from the EastStack stack. Therefore, since a is above b and b is above c, then b must be to the left of both a and c because of the clear half-corridor property. We first rule out the possibility that ab or bc are class III edges that are derived from the WestStack. Suppose, to the contrary, that ab is a class III edge derived from the WestStack. This implies the existence of an object in S, call it o, such that o intersects with $\textsc{South}^-(a) \cap \textsc{North}^-(b)$, and $\textsc{West}^-(o)$ has a non-empty intersection with both a and b. This implies that o intersects with $\textsc{East}^-(a) \cap \textsc{South}^-(a) \cap \textsc{North}^-(b)$ because the set is axis oriented. On the other hand since ac is a class III edge the clear half corridor implies that no point of o can lie in $\textsc{East}^-(a) \cap \textsc{South}^-(a) \cap \textsc{North}^-(b)$, thus realizing the desired contradiction.

Similarly bc cannot be a class III edge derived from the WestStack.

Now assume that centre(d) is interior to $\triangle(abc)$. Then as in case 1. above we see that b cannot simultaneously see both a and c with class I edges. The clear corridor property precludes the possibility that b sees a and c simultaneously with class II edges. The clear half corridor property precludes b seeing a and c simultaneously with class III edges that are derived from the EastStack. In fact it is easy to see that no combination of class I, class II, or class III edges can be used so that b simultaneously sees a and c when centre(d) is inside $\triangle(abc)$. Therefore, we have obtained the desired contradiction.

Through a detailed case analysis we have established that no triangle in M is a separating triangle. $\square$

Recall that we defined a QNST as a triangulation with no separating triangles, and four vertices on the boundary of the unbounded face. We now show how M can be augmented with four additional vertices to obtain a QNST. We can describe the outer face of M by four paths. Beginning at east(S) and proceeding in a counter clockwise direction we have, $P_1 = [\text{east}(S), \ldots, \text{north}(S)], P_2 = [\text{north}(S), \ldots, \text{west}(S)]$, $P_3 = [\text{west}(S), \ldots, \text{south}(S)]$, $P_4 = [\text{south}(S), \ldots, \text{east}(S)]$. Observe that these paths are overlapping, and all contain at least one vertex. Let R denote a graph consisting of M and four additional vertices labeled $r_1, \ldots, r_4$, and the edges $\{p_i r_i\colon\ p_i \in P_i,\ i = 1, \ldots, 4\} \cup \{r_i r_{i+1},\ i = 1, \ldots, 3\} \cup \{r_4 r_1\}$. In our example in Fig. 6 we have $P_1 = [5, 1, 0]$, $P_2 = [0, 3]$, $P_3 = [3, 6, 8, 9]$, and $P_4 = [9, 5]$.

Given an object $s \in S$ we say that s is *first quadrant maximal*, if the region $\textsc{North}^-(s) \cap \textsc{East}^-(s)$ is in the complement of S. We can similarly define maximality with respect to the other quadrants. An important property of objects on P_i, is that they are maximal, with respect to quadrant i, as we prove in the following lemma.

Lemma 3.7. *Let S be a non-aligned axis-oriented set, and let $s \in S$ be an element of P_i, then s is maximal with respect to quadrant i.*

Proof. For concreteness and ease of exposition, we argue the case for the path P_1. When P_1 has one or two elements, then the lemma holds, because both $\textsc{North}(S)$ and $\textsc{East}(S)$ are first quadrant maximal. For paths with at least three elements we show that the existence of an object $a \in P_1$ such that a is not first quadrant maximal leads to a contradiction. Consider such an a. Then there exists an object, b in P_1, that is right and above a. This implies that when traversing P_1 between a and b we encounter a consecutive triple s above r above t such that both s and t are to the right of r. However this implies that there is a class III edge derived from the EastStack between s and t, a contradiction. Therefore all elements in P_1 are first quadrant maximal, and similarly elements in P_i are maximal with respect to quadrant i. □

We can now prove that our final triangulation is a QNST triangulation.

Lemma 3.8. *R is a QNST triangulation.*

Proof. The outer face of R is clearly a quadrangle, and R is a triangulation. We have already shown that M is an NST triangulation. If we can show that there are no chords on any of the paths P_i, then this implies that there are no separating triangles in R. Consider two non-consecutive objects on P_i, $i = 1, \ldots, 4$, and assume that there is an edge, e, between them in R. For concreteness suppose $i = 1$ thus the edges in P_1 are all from class III derived by the EastStack. The edge e obviously cannot be from class II, and by an argument similar to the one given in case 3 of Lemma 3.6 it cannot be a class III edge. So suppose e is a class I edge between objects s and t, and r is an object in between s and t in P_1, such that s is above r and r is above t. However, if s sees t by a class I edge the axis-oriented property implies that both s and t are to the left of r, contradicting the fact that t is first quadrant maximal. Thus R is an NST triangulation. □

We have so far limited our attention to an axis-oriented set that is not axis-aligned. Our main motivation for doing this was to simplify the exposition. It turns out that we can still apply our results, even if the axis-oriented set does exhibit pairs of object that are axis aligned. We can rotate our coordinate system by a small angle, so that no two objects are axis aligned, no new class I visibility edges are added, and the same relative ordering with respect to the compass directions is retained for object pairs. This maintains the visibility required when assigning class III edges. It also ensures that the objects are correctly ordered, precluding chords on the outer boundary paths $P_1 \ldots P_4$.

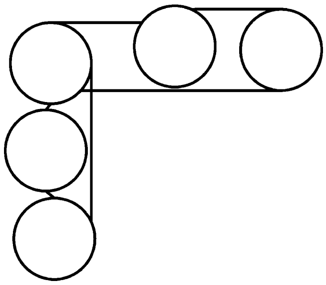

Fig. 7. Lines of sight that realize the triangulation may cross.

There is actually no need to explicitly compute the small angle of rotation, because we can simulate its effect. Recall that in algorithm Make-M we sweep a horizontal sweeping line across the plane from the top to the bottom of our set of objects. The non-alignment assumptions imply that no two objects have the same NORTH supporting line. In the case where we do encounter an axis alignment we can simulate the effect of sweeping a line that is slightly tilted upwards on the right. Thus for two objects supported from above by the same horizontal supporting line, the one to the left is encountered first by the tilted supporting line. Similarly, when determining the relative ordering in a vertical direction, say EAST, we can assume that the vertical axis is slightly tilted as well, so that for two objects with the same vertical right support lines the lower one is encountered first. In this way we treat the objects as if the coordinate system does not cause axis alignment.

We can now state our first main result.

Theorem 3.9. *Given an axis-oriented set, S, we can always obtain a Hamiltonian path using lines of sight. Furthermore, using the lines of sight we can obtain a perfect matching of the objects with non-crossing lines of sight.*

Proof. We construct the NST triangulation M as above. If the outer face of M is of cardinality four or less then M is Hamiltonian. Otherwise, we augment M to obtain the graph R that is QNST. Since there is always a Hamiltonian cycle in R that uses three edges on the outer face of R consecutively, we also have a Hamiltonian path through the graph M. Observe that the lines of sight in the path may cross as shown by the small example given in Fig. 7. However, since we can obtain a planar drawing of M, realizing graph edges by joining object centres, we see that any two lines of sight that do cross must emanate from the same object. Therefore, we can use alternate edges in a Hamiltonian path to obtain a crossing free matching. □

4. Convex translates

In this section we show that by applying a suitable affine transformation to a set of translates of the same compact convex object in $\mathbb{R}^2$ we can use our methods to

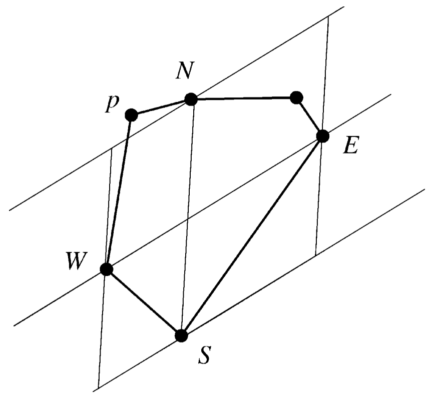

Fig. 8. The triangle E, W, p has greater area than triangle E, W, N.

obtain Hamiltonian paths and matchings in the visibility graph of the objects. It will be convenient to retain our orthogonal axes influenced terminology. Thus, in the sequel we assume that we have an arbitrary underlying coordinate system, with two non-collinear axes. One of them will be called horizontal and the other vertical.

The following result was communicated to us by W. Kuperberg, who attributes the proof to "folklore". An alternate proof of this result can be found in [11].

Lemma 4.1. *Let x be a compact convex subset of the plane, that is neither a triangle nor a line segment. There exists four extreme points of x, N, E, S, W not all on the same line, and a coordinate system where the line through N, S is vertical, and the line through E, W is horizontal, such that N and S are contained in horizontal lines of support, and E and W are contained in vertical lines of support.*

Proof. Consider a largest area 4-gon inscribed in x, and label its vertices N, E, S, W in the obvious way. We call the line through N, S vertical and the line through E, W horizontal. We show that the points N, E, S, W are respectively consistent with the designations north(x), east(x), south(x), west(x). We argue that N is contained in a horizontal support line. The consistency of the designations for the other extreme points can be shown in a similar way.

Let us assume to the contrary that the horizontal line passing through N is not a support line for x, thus there is a point on the boundary of x, call it p, that is in the half-plane $\text{N}\small{\text{ORTH}}^-(x)$. See Fig. 8. However, we can then construct a triangle E, W, p of greater area than the triangle E, W, N, thus contradicting the initial assumption that N, E, S, W are the corners of a largest area 4-gon inscribed in x. $\square$

A line segment is axis oriented, if it is parallel to one of the coordinate axes. A triangle is axis oriented if one coordinate axis is parallel to its base and the other axis is parallel to any line intersecting the base and passing through the apex of the triangle. As a consequence of the previous remark, and the previous lemma we conclude that there exists an affine transformation to convert any compact convex object in $\mathbb{R}^2$ into an axis-oriented object. Therefore, there is also a transformation to obtain an axis-oriented set from a set of convex translates. We conclude with the second main result of our paper.

Theorem 4.2. *Given a set of translates of a compact convex object in $\mathbb{R}^2$, we can always obtain a Hamiltonian path using lines of sight. Furthermore, using the lines of sight we can obtain a perfect matching of the objects with non-crossing lines of sight.*

5. Discussion

We have shown that the visibility graph of a set of translates of a compact convex object always admits a Hamiltonian path, and a perfect matching can be found by using non-crossing lines of sights. Our notion of visibility is described by in O'Rourke's book [9] as *clear visibility*. In [9] the line of sight may have grazing with an object. This condition is strictly weaker than clear visibility, so our results apply even if grazing contact is permitted.

Our proof is constructive, and does not explicitly use the visibility graph. Consider a set S of n objects. The graph that we construct has complexity in $O(n)$, whereas the visibility graph of S may be a complete graph, that is it may have $\binom{n}{2}$ edges. Furthermore, we can obtain a Hamiltonian path and perfect matching in $o(n^2)$ time.

We briefly sketch an approach that results in an algorithm with complexity $O(n\log n+k)$ for a set of n, k-gons. We use standard computational geometry techniques, as described in [9] or [2].

The first step in our algorithm finds a coordinate system so that the objects are axis oriented. Given a convex polygon with k vertices the largest area inscribed 4-gon can be found in $O(k)$ time by using the method of rotating calipers see [9]. Briefly, this technique simulates rotating the legs of calipers tangent to the boundary of the convex polygon. In our case we use two sets of calipers. Once an initial positioning is found we can proceed by rotating both calipers in the same direction. Since a convex k-gon has $O(k)$ antipodal pairs [9] (pairs of points admitting parallel lines of support) and we can proceed from one configuration of calipers to the next in constant time, an $O(k)$ time algorithm finds the largest inscribed 4-gon. See Fig. 9.

Subsequently we can apply our sweeping line algorithm. It is a routine matter to show that the running time of our algorithm Make-M is in $O(n\log n)$. We can then augment the graph M, to get the QNST triangulation R, in $O(n)$ time.

Given a QNST triangulation, a Hamiltonian path can be found in linear time as described by Dillencourt [5] by using the algorithm of Asano et al. [1]. Obtaining a perfect matching from the Hamiltonian path is a trivial matter. Thus we can complete the whole process in $O(n\log n + k)$ time.

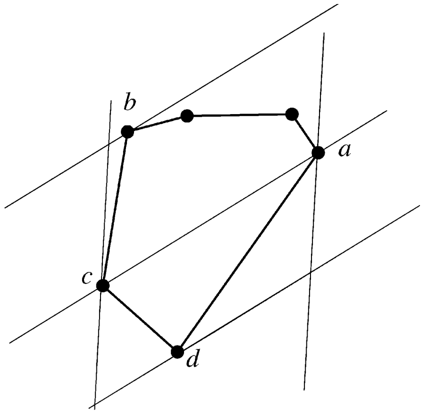

Fig. 9. Two rotating calipers used to find the largest area 4-gon. Observe that the direction of the line through ac determines the pair bd.

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