COMPUTING SIMPLE CIRCUITS FROM A SET OF LINE SEGMENTS IS NP-COMPLETE

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Abstract

Given a collection of line segments in the plane we would like to connect the segments by their endpoints to construct a simple circuit. (A simple circuit is the boundary of a simple polygon.) However, there are collections of line segments where this cannot be done. In this note it is proved that deciding whether a set of line segments admits a simple circuit is NP-complete. Deciding whether a set of horizontal line segments can be connected with horizontal and vertical line segments to construct an orthogonal simple circuit is also shown to be NP-complete.

1. Introduction

A natural generalization of the problem of finding simple circuits from a set of points, is the problem of finding a simple circuit from a set of line segments. In general, a set of line segments does not necessarily admit a simple circuit. An example of a set of line segments that does not admit a simple circuit is given in figure 1. An interesting question is: When do a set of line segments admit a simple circuit?

The task of obtaining a simple circuit from a set of points is a recurring theme that appears in a variety of applications. In network routing problems, the tour of shortest Euclidean distance that begins and ends at a common site, and visits all other sites exactly once, is a simple circuit. This is the Euclidean travelling salesman problem, and is known to be NP-complete [GJT,IPS,LLRKS]. Simple circuits have also been used in the area of pattern recognition, for extracting perceptual information from sparse data [M,O'RBW,T1,T2]. Surprisingly, the problem of computing simple circuits from a set of line segments has not received much attention.

Given a collection of sites on a plane, and the requirement that certain sites are visited in a prescribed order, the task of finding a shortest tour reduces to finding shortest tours from a set of line segments, or chains of line segments. If in addition, edges in the tour that cross greatly increases the cost of the tour (because bridges or tunnels are required), then one is interested in finding a shortest tour with few crossings. In the extreme, this reduces to deciding if a set of line segments, or chains of line segments, admits a simple circuit. In the context of pattern recognition, meaningful perceptual information may be obtained from a collection of line segments, or chains of line segments, by computing a minimal set of disjoint simple circuits. Again, in the extreme, this problem reduces to deciding if a set of line segments admits one simple circuit.

In [RIT] it was shown that if the set of line segments are constrained so that every segment has at least one of its endpoints on the convex hull of the segments, an O(*nlogn*) algorithm can be used to determine whether the set admits a simple circuit. Furthermore, one can deliver simple circuits, and even optimize over the area and perimeter of the polygons constructed, in the same time bound. Other special cases of the problem of obtaining a simple circuit from a set of line segments are discussed in [R2].

In this note, it is shown that to determine whether a set of segments admits a simple circuit is NP-complete. After preliminary definitions a reduction is given to a variant of this problem. In this variant we are given a set of orthogonal line segments, and are required to construct a simple circuit whose edges are all orthogonal to the coordinate axes. It is then shown how to reduce the orthogonal simple circuit problem to the simple circuit problem.

2. Preliminaries

A *simple circuit* is a sequence of edges, $e_0, e_1, \ldots, e_k$, such that for all $0 \le i \le k$, e_i and $e_{(i+1) \bmod k}$ intersect at their endpoints, and no other intersections between edges occur. A simple circuit is the boundary of a simple polygon. Let S be a set of line segments in the plane. We require that the segments be properly disjoint, that is, no segments intersect in their interiors. However, we permit segments of S to intersect at their endpoints. If a set A can be found such that $S \cup A$ is a simple circuit of $|S| + |A|$ edges, then we say that S *admits* a simple circuit and A is a set of *augmenting segments*. To obtain a set of augmenting segments, we begin by considering a set of segments as *candidates*. We say that two points *see* each other, if the line segment between them does not intersect any segment. Since we are looking for crossing free circuits, it is natural to choose as candidates connections of endpoints of segments that see each other.

An *orthogonal simple circuit* is a simple circuit whose edges are orthogonal to the coordinate axes. One can say two points *orthogonally see* each other, if they agree in one of their coordinates and they see each other.

It will be shown that the following problem is NP-complete.

Simple Circuit (SC)

INSTANCE: A set of line segments S.

QUESTION: Does S admit a simple circuit?

To simplify the presentation it will first be shown that the following more structured problem is NP-complete.

Orthogonal Simple Circuit (OSC)

INSTANCE: A set of line segments S orthogonal with respect to the coordinate axes.

QUESTION: Does S admit an orthogonal simple circuit?

3. Orthogonal Simple Circuits is NP-complete.

In this section it will be shown that the orthogonal simple circuit problem (OSC) is NP-complete. The following problem is known to be NP-complete [GJT].

Hamiltonian Path in Planar Cubic Graphs (HPPCG)

INSTANCE: A planar cubic (all vertices are of degree three) graph G=(V,E).

QUESTION: Is there a Hamiltonian path in G?

The idea behind the transformation of HPPCG to OSC is to build *modules* out of collections of line segments. Given a planar cubic graph G, the collection of modules M(G) is constructed. Each module m_a will uniquely represent a vertex a of G. The edges of the graph will be simulated by a subset of the candidates of the collection of modules. The remainder of this section will lead to the conclusion that a Hamiltonian path exists in a planar cubic graph if and only if the set of corresponding modules admits an orthogonal simple circuit.

To obtain modules from vertices, it is necessary to first compute a *rectilinear planar layout* of the graph. This layout maps vertices to horizontal line segments and maps edges to vertical line segments, with all endpoints of segments at positive integer coordinates. Two horizontal segments are intersected by the endpoints of a vertical segment, if and only if the corresponding vertices are adjacent in the graph. Figure 2 shows a straight line drawing of a planar cubic graph with its rectilinear planar layout.

In [RT] it has been shown that a rectilinear planar layout can be computed for planar graphs with n vertices in $O(n)$ time. The height of the layout of this algorithm is guaranteed to be at most n, and the width at most F, where F is the number of faces in the graph. In cubic graphs $F = n/2 + 2$ by Eulers relation. For details regarding the algorithm used to obtain rectilinear planar layouts, refer to [RT].

Given a planar cubic graph G=(V,E), modules are built using the following procedure:

ALGORITHM CONSTRUCT MODULES

1. Obtain a rectilinear planar layout of G.

2. **for** each vertex v in G **do**

   Let (x_l, y), (x_r, y) denote the coordinates of the horizontal line segment h that corresponds to v. Every vertex in G is of degree three. Therefore, there are three vertical segments intersecting h. Denote their intersection points as (x_1, y), (x_2, y) and (x_3, y). **if** x_i is the top endpoint of its vertical line segment

   then $top_i \leftarrow$ true **else** $top_i \leftarrow$ false.

   2.1. Construct an outer rectangle with diameter, $(6x_l, 4y)$, $(6(x_r+1)-1, 4y+3)$ and an inner rectangle with diameter, $(6x_l+1, 4y+1)$, $(6(x_r+1)-2, 4y+2)$.

2.2. **for** $i \leftarrow 1$ **to** 3 **do**
if top_i is true **then**
Place a gap of width one at $(6x_i+2,4y)$ of the outer rectangle and at $(6x_i+2,4y+1)$ of the inner rectangle.
else
Place a gap of width one at $(6x_i+2,4y+3)$ of the outer rectangle and at $(6x_i+2,4y+2)$ of the inner rectangle.

The graph in figure 2, yields the modules shown in figure 3.

The gaps in the outer rectangular frame will be denoted as *doors*. The segments of the inner rectangular frame will be denoted as *enforcers*, because this most accurately reflects their purpose. The nature of their enforcement will become clear later in the discussion. If the edge (a,b) is in G, then a door of module a faces a door of module b. We say that these doors are *neighbours*. By a logical extension, modules a and b are also termed as neighbours, if their doors are neighbours. Observe that all modules are topologically equivalent.

There is a relationship between Hamiltonian paths in planar cubic graphs and orthogonal simple circuits in the modules just constructed. Suppose we are given a planar cubic graph and a Hamiltonian path in the graph, where we label the vertices 1, ..., n representing the permutation that corresponds to the Hamiltonian path. It will be shown how to obtain an orthogonal simple circuit from the corresponding configuration of modules. Since for every vertex in a planar cubic graph there is a unique module, we can label modules the same way we have labeled the vertices. For every edge, $(i,i+1)$, $i = 1,..., n-1$, in the Hamiltonian path, we connect the doors of module i to module $i+1$. Doors that are not used to connect neighbours are connected to their enforcers. There is one exception; the modules corresponding to 1 and n, the terminal vertices in the Hamiltonian path, have one of their unused doors closed. Finally, the remaining augmenting segments connecting enforcers are now uniquely defined. A collection of modules, connected as prescribed above is shown in figure 4, where the path of modules 1,2,4,3,7,6,5,8 is a simple circuit. To verify that an orthogonal simple circuit is obtained, notice that each module (except 1 and n) has two simple paths running through it. These two paths run through the entire network of modules until the terminal modules are reached, that is, modules 1 and n. These modules have a single orthogonal simple path running through them, and in turn, complete the two previously mentioned paths.

Let a *path of modules* denote a sequence of modules $m_1, m_2 \cdots, m_k$ where, a) m_i is a neighbour of m_{i+1}, for $1 \le i \le k-1$, b) m_i is connected to m_{i+1} with two augmenting segments, and c) enforcers are connected as described above.

Lemma 1: A path of modules is an orthogonal simple circuit.

Proof: Follows directly from the preceding discussion. □

Lemma 2: Given a planar cubic graph, G, and a permutation of its vertices representing a Hamiltonian path, then an orthogonal simple circuit can be obtained in a collection of modules, M(G), in polynomial time.

Proof: Construct the path of modules according to the prescribed permutation. This can be done in linear time. □

It remains to show that every orthogonal simple circuit in M(G) can be used to obtain a Hamiltonian path in G in polynomial time.

Let us examine the ways in which a module can be connected to its neighbouring module with augmenting segments. A first step is to establish for each module a list of candidates. In figure 5, a module is shown with its entire set of candidates drawn in dotted lines.

The following lemma exhibits a crucial property of two neighbouring modules.

Lemma 3: If two neighbouring modules are connected by a single augmenting segment, then an orthogonal simple circuit cannot be obtained in the collection of modules.

Proof: First observe that all doors of modules have a similar structure. There are cases where there are two doors that are in the same vertical row. This is the case that will be argued. In the simpler case (for example the unlabeled door in figure 5) a similar but simpler argument is required. Referring to figure 5, assume we have the augmenting segment (7,x). The edge (8,6) is forced, because we are assuming there is only one augmenting segment between modules. Now 5 can only be connected to 3, which causes a disjoint circuit. This rules out the possibility of getting a single orthogonal simple circuit. Assume instead that we have augmenting segment (8,y). This forces (7,5) and (6,4) which forces (3,1). But this causes a disjoint circuit. Therefore, neighbouring modules are connected with two augmenting segments, in every orthogonal simple circuit . □

We have established that at each door modules are connected with two augmenting segments. Therefore, saying two modules are *connected*, refers to the inclusion of the two augmenting segments between the doors of modules. Another fact that is necessary is:

Lemma 4: A module can be connected to at most two of its neighbours.

Proof: Assume that a module is connected to three of its neighbours. This leaves the enforcers disconnected from the rest of the segments of the module and isolated from all other modules. □

It should now be clear how the internal structure used for each module acts as an enforcer. It forces each door to have two augmenting segments connecting each neighbour, and it forces each module to be connected to at most two of its neighbours. In the construction that led to lemma 1, it was shown that a module can be connected to one or two of its neighbours in an orthogonal simple circuit.

It is obvious that every module must be connected to at least one other module in every orthogonal simple circuit. Let the *circuit degree* of a module denote the number of neighbours a module is connected to in an orthogonal simple circuit.

Lemma 5: Every orthogonal simple circuit through a collection of modules is a path of modules, that is, it has exactly two modules of circuit degree one, and all other modules of circuit degree two.

Proof: We cannot have an odd number of modules with circuit degree one. To see this, first obtain the total sum σ of all circuit degrees. If there is an odd number of modules of circuit degree one, then σ is odd. But σ must always be even, since σ counts each module connection twice.

Suppose there are four or more modules with circuit degree one. Since each module can have circuit degree at most two, σ is at most $2(n-4)+4$. As in graphs, if the sum of the degrees is less than $2n$ (less than n edges), then the graph must have disconnected components. Similarly with σ less than $2n$ there must be some disconnected modules.

Suppose there are no modules of circuit degree one. Therefore, all modules are of circuit degree two. It has been shown in lemma 1 that a path of modules is a simple circuit. If all modules are of circuit degree two, then we have the equivalent of a path of modules that is connected at its endpoints (module m_1 is connected to m_k.) This is topologically equivalent to taking a simple circuit, breaking it into two disjoint paths, and connecting each path to itself, thus creating two disjoint circuits.

Therefore, every orthogonal simple circuit must be a path of modules. □

Finally, the preceding lemmas lead to the conclusion:

Theorem: OSC is NP-complete.

Proof: It is routine to show that OSC is in NP. If we are given a set of orthogonal segments with a set of augmenting segments, then the existence of a simple orthogonal circuit can be checked in linear time. We have shown that given a planar cubic graph, G, we can construct a a collection of modules, M(G), such that there is a Hamiltonian path in G, if and only if there is an orthogonal simple circuit in M(G). Therefore OSC is NP-complete. □

In the reduction that has just been given there are segments that intersect at their endpoints. It is not hard to convert the collection of modules into a set of disjoint horizontal segments. Simply remove the vertical edges of each module, and place each module on a row of its own. Modules with these changes are shown in figure 6. By examination one can see that all the vertical segments that were removed are now forced in the new layout of horizontal segments.

It is worth noting at this point that a remarkably similar problem has a polynomial time solution. Suppose we are given a set of orthogonal line segments, and we wish to determine if the segments admit an *alternating orthogonal simple circuit*. This restricts the resulting orthogonal simple circuit to have edges that alternate between horizontal and vertical. An algorithm due to O' Rourke [O'R], to decide whether there is an alternating orthogonal simple circuit in a set of points, can be applied in a straightforward manner. This algorithm returns an orthogonal simple circuit, if it exists, in O(*nlogn*) time. Furthermore, it is shown that if an alternating orthogonal simple circuit exists, then it is unique. Concerning (not necessarily alternating) orthogonal simple circuits of points, this problem has been shown to be NP-complete [R1,R2].

4. Simple Circuit is NP-complete

In this section it will be shown that OSC polynomially transforms to SC. Following the strategy taken in the previous section, we will build a collection of modules out of line segments. As a distinguishing feature we will denote the modules described in this section as SC modules, and those of the previous section as OSC modules.

SC modules are constructed in much the same way as OSC modules. Each module has an inner and outer rectangular frame with doors on the outer frame. Shown in figure 7, is a collection of SC modules corresponding to OSC modules in figure 3, and ultimately to the graph in figure 2. In SC modules greater care must be taken to limit the visibility of doors. For this reason the doors are recessed. Each door in an SC module is one unit wide, in a recessed three unit enclosure that is two units deep. See figure 7. This limits the field of view of any door to an angle of forty five degrees. Let D be the distance between the top of the lowest module and the bottom of the highest module. If h is the total number of rows in the rectilinear planar layout (h is bounded by n, the number of vertices in G), and each module is w units wide (SC modules are constructed with a width of 10 units), with a one unit space between rows of modules, then $D = (h-2)(w+1)+1$. Doors are spaced so that the distance between them is at least D. The limited field of view ensures that a door can only see its proper neighbour. Another feature found in SC modules, and not in OSC modules, is the obstacle that runs the length of every module. These obstacles ensure that the visibility between a door and its enforcer remains local. The details concerning the construction of OSC modules are omitted, as they are quite tedious, but given the above informal description it is a routine matter to construct the SC modules in $O(n)$ time.

A correspondence between the candidates of an OSC module and an SC module will be established. Since there are some candidates (endpoints that see each other) in SC modules that do not exist in OSC modules, it will be necessary to introduce the notion of a *useful candidate*. A candidate is useful if it can eventually appear as an augmenting segment. As will be shown, there are some candidates in SC modules that are not useful. However, all useful candidates in SC modules correspond exactly to the useful candidates of an OSC module.

Referring to figure 7, there are candidates in OSC modules that cross. For example the candidates (5,6) and (6,7). It will be shown that all these crossing candidates are not useful. Since the candidates found at every door of every module are equivalent, it is sufficient to examine a single door. Referring to figure 7, the inclusion of (5,8) forces (6,7), a crossing. Similarly (6,7) forces (5,8), a crossing. Therefore, (6,7) and (5,8) can never be augmenting segments. Including (7,y) forces either (8,x), (8,6) or (8,5). With (8,x) we get a crossing, with (8,6) we isolate 5, and (8,5) = (5,8) was shown to be forbidden. As is the case for (7,y), (8,x) can never appear in a simple circuit. Therefore, (5,8), (6,7), (7,y) and (8,x) cannot be augmenting segments.

We can now conclude that the list of useful candidates in OSC modules is identical to those in SC modules. It is not hard to see that OSC modules are topologically equivalent to SC modules. Therefore:

Theorem: There is an orthogonal simple circuit in OSC modules if and only if there is a simple circuit in the corresponding SC modules.

The main result of the paper can now be proved.

Theorem: SC is NP-complete.

Proof: It is routine to verify that SC is in NP. Since OSC is NP-complete the previous theorem implies that SC is also NP-complete. □

In the previous section it was shown that OSC modules could be built from individual disjoint line segments. This does not appear to be the case for the SC modules described here. It remains an open problem to determine if SC is NP-complete even if we insist that all segments are disjoint.

Observe that all the segments used in this reduction for SC are orthogonal. Therefore, SC is NP-complete for orthogonal segments.

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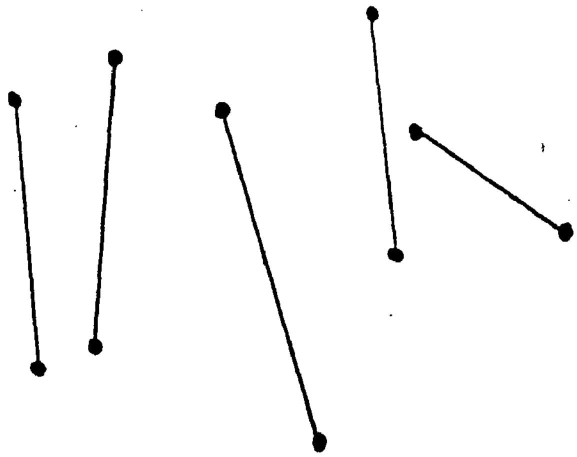

Figure 1

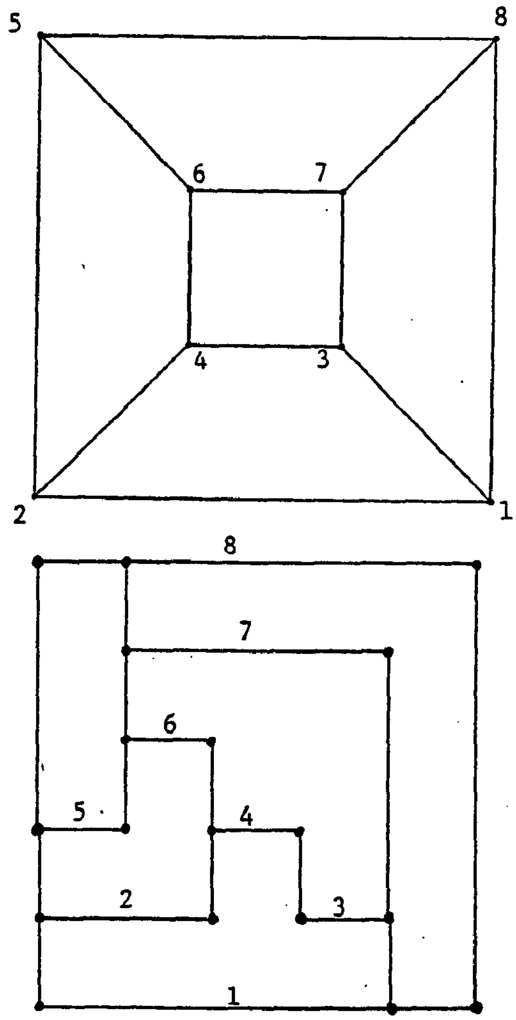

Figure 2

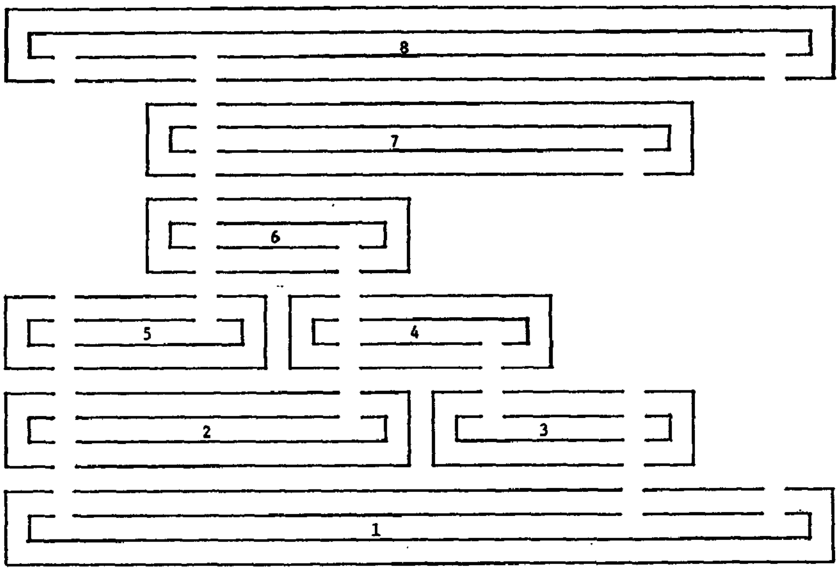

Figure 3

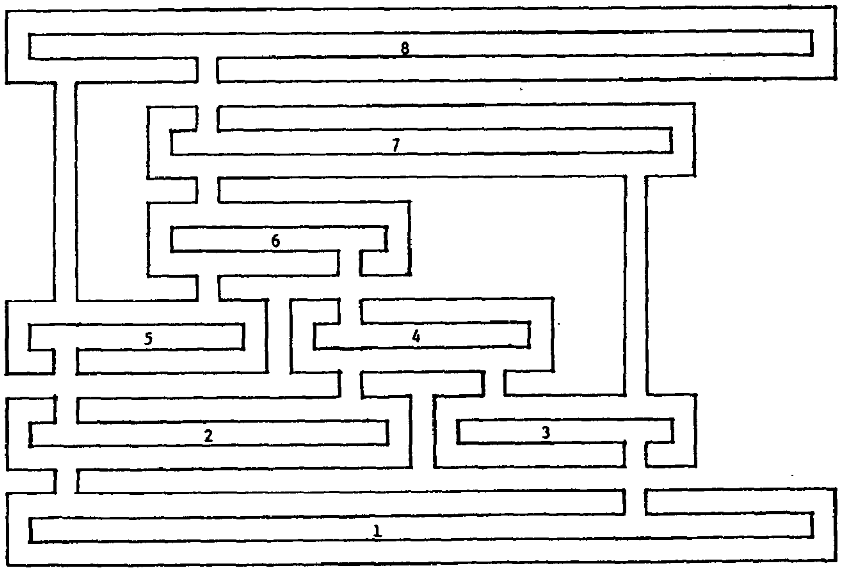

Figure 4

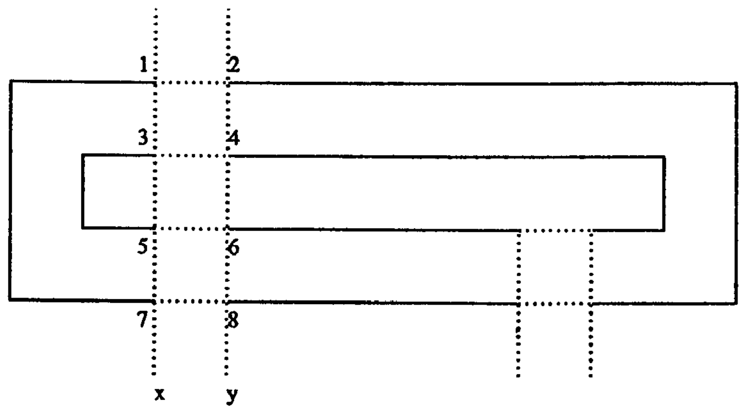

Figure 5

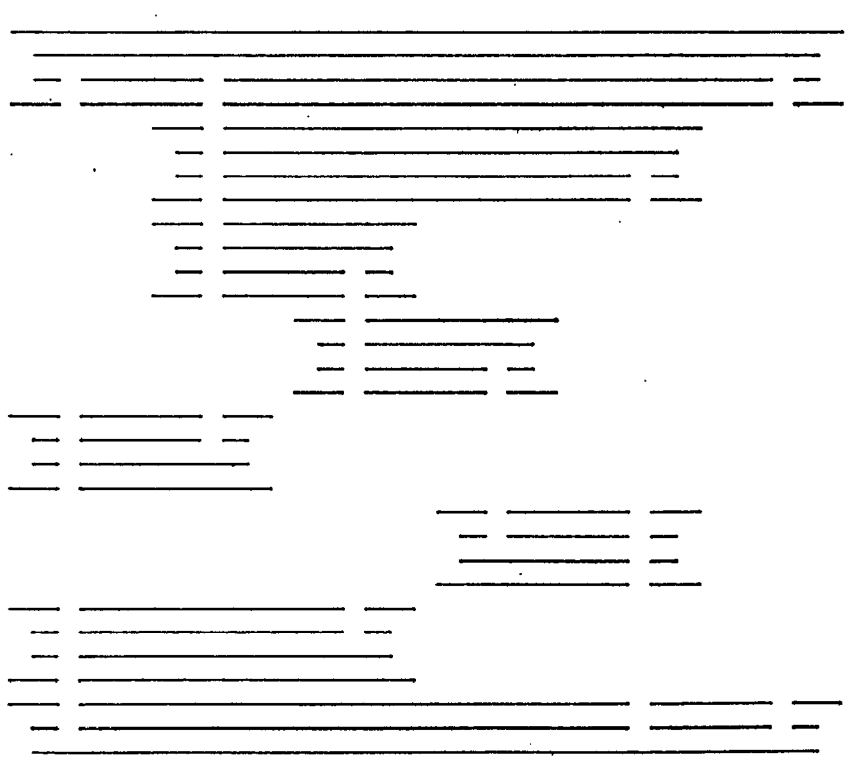

Figure 6

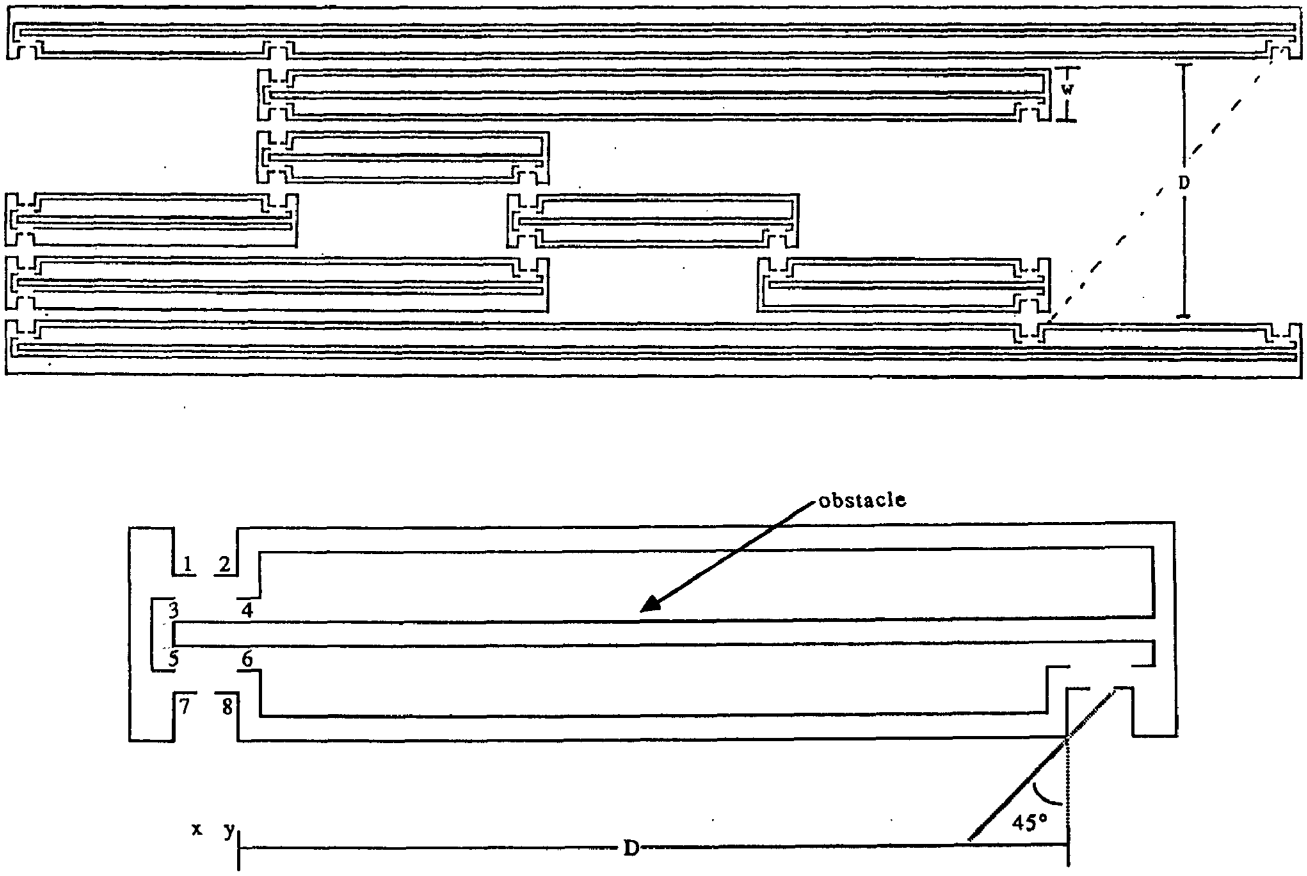

Figure 7