# Hamilton Circuits in Hexagonal Grid Graphs

Kamrul Islam(1) Henk Meijer(1,2) Yurai Núñez(1) David Rappaport(1) Henry Xiao(1)
(1)Queen's University, Canada (islam/yurai/daver/xiao@cs.queensu.ca)
(2)Roosevelt Academy, The Netherlands (h.meijer@roac.nl)

## Abstract

We look at a variant of the Hamilton circuit problem, where the input is restricted to hexagonal grid graphs. A hexagonal grid graph has a vertex set that is a subset of the grid points of a regular hexagonal tiling of the plane and edges corresponding to hexagon sides. We show that Hamilton circuit in hexagonal grid graphs is NP-complete.

## 1 Introduction

Optimization problems have been at the core of computing since the advent of the computer age more than half a century ago. One of the most prominent optimization problems, if not the most prominent of them all, is the Travelling Salesman Problem (TSP) . Simply put, this problem tries to find a shortest cyclic tour that visits all nodes of a network. It is well known that the TSP is NP-Hard, (see the classic book by Garey and Johnson which continues to be the standard reference on the subject [1]) and that the reduction is from the Hamilton circuit problem.

With regards to the TSP a natural question is the complexity of the problem where the edge weights of the graph are Euclidean distances. This poses a severe technical impediment as irrational Euclidean lengths are not computable. Nevertheless, this problem was addressed by considering a geometric graph, a *grid graph*, whose vertex set is a subset of the grid points of a unit square tiling of the plane, and whose edge set connects vertices that are grid points that are one unit apart. Thus all Hamilton circuits in a grid graph with $n$ vertices have Euclidean length exactly $n$. Hamilton circuits in grid graphs was shown to be NP-complete by Itai, Papadimitriou and Szwarcfiter [2]. In [2] a question is raised regarding special cases of grid graphs where the Hamilton circuit problem becomes solvable in polynomial time. The most obvious special case is for *solid grid graphs*, that is, a grid graph with no internal face containing a grid point in its interior. This question has since been settled by Umans and Lenhart [7] who gave a polynomial time algorithm for the Hamilton circuit problem on a class of graphs that is a superset of solid grid graphs.

Subsequently Polishchuk, Arkin, and Mitchell [4] have studied the Hamilton circuit problem for triangular grid graphs. A *triangular grid graph* has vertices that are a subset of grid points of a unit edge length regular triangular tiling of the plane, and whose edge set connects vertices that are one unit apart. Polichuk et. al. prove that in general the Hamilton circuit problem for triangular grid graphs is NP-complete. They also show that for solid triangular grid graphs the Hamilton circuit problem is solvable in polynomial time. In fact, except for one special case all solid triangular grid graphs are Hamiltonian.

A natural question arising from the results in [4] is to consider the remaining regular tiling of the plane, that is, tiling with hexagons. A hexagonal grid graph has a vertex set that is a subset of the grid points of a unit side length regular hexagonal tiling of the plane, and whose edge set connects vertices that are one unit apart. In this paper we show that Hamilton circuit is NP-complete for hexagonal grid graphs.

## 2 Hamilton Circuits in Hexagonal Grid Graphs is NP-Complete

In this section we will prove that the following problem is NP-complete.

[ HCH ] **Hamilton Circuits in Hexagonal Grid Graphs**
INSTANCE: A Hexagonal Grid Graph $H$.
QUESTION: Is there a Hamilton Circuit in $H$?

To prove the NP-completeness of HCH we will perform a polynomial reduction from the following problem which is known to be NP-complete [3].

[ HCB ] **Hamilton Circuits in Planar Bipartite Graphs**
INSTANCE: A planar 2-connected bipartite graph $G$ with maximum degree 3.
QUESTION: Is there a Hamilton Circuit in $G$?

We will prove the following theorem.

**Theorem 1** *Given an instance of HCB, $G$, there is a polynomial transformation of $G$ to an instance of HCH, $H$, such that $G$ is a yes instance if and only if $H$ is a yes instance.*

We begin the proof by providing a polynomial transformation from an instance $G$ of HCB to an instance $H$ of HCH. The fact that $G$ is bipartite is an essential ingredient in our transformation. To distinguish between vertices in separate bipartitions of $G$ we use the terms *light* and *dark* grey vertices. The transformation can be outlined with the following process.

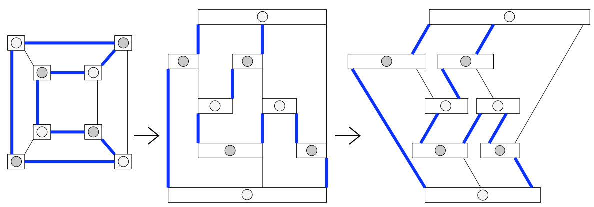

Figure 1: We show from left to right, a planar digraph with maximum vertex degree 3, $G$, a planar rectilinear layout of $G$, and its drawing $D(G)$.

**Transformation T**

1. Given $G$ we obtain a drawing $D(G)$ of $G$ as shown in Figure 1.

2. We distinguish several elements of $D(G)$, that is, the light and dark vertices, and the edges between them. For each of these elements we provide a hexagonal grid graph that acts as a gadget that simulates the element of $G$.

3. We show how to combine the gadgets culminating in the desired instance $H$, of HCH.

The details of transformation $T$ follow.

Given a planar bipartite graph with maximum vertex degree 3, $G$, we can obtain a rectilinear configuration using the methods of Rosenstiehl and Tarjan [5] and Tamassia and Tollis [6]. We modify the rectilinear drawing slightly to obtain a drawing of $G$, $D(G)$, that leads to a hexagonal grid graph simulation of $G$, the graph $H$. In Figure 1 we show an example graph $G$ and its drawing as a planar rectilinear layout, and the drawing $D(G)$. The vertices in the drawing $D(G)$ are represented by horizontal bars. The edges of the drawing are one of two fixed angles, 60 or 120 degrees. This drawing is based on the so called *st*-ordering of the vertices of a planar graph. In an *st*-ordering we can choose two vertices of a face (the external face) and designate $s$ and $t$ the unique source and sink of a topological ordering of the graph. This ordering implies a directed acyclic structure with a single source and sink, which in the drawing goes from top to bottom. We choose both

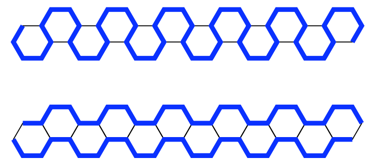

Figure 2: We show a strip and with bold lines how it can be traversed with a sequential and parallel paths.

$s$ and $t$ to be light vertices, as this will simplify the transformation from $D(G)$ to $H$. All vertices, except the vertices $s$ and $t$, have at most two upward or downward edges. The two terminal vertices, $s$ and $t$, have all edges going in the same direction. The ordering of the edges from left to right is obtained by a compatible *st*-ordering, of the dual graph of $G$ as shown in [5, 6]. We can convert the rectilinear planar drawing to $D(G)$ by a left to right sweep of the edges. In this way we can draw 60 and 120 degree edges and maintain planarity.

Edges of $D(G)$ are simulated in the hexagonal grid graph $H$ by *strips*. Strips come in three varieties distinguished by the counter-clockwise angle made with the x-axis, 0, 60, and 120 degrees. The 60 and 120 degree strips are used to simulate edges. We call a 0 degree strip internal to a vertex gadget an *extender* as it is used to simulate the extent of the width of a bar from one end to the other. We use double arrows in Figure 4 to illustrate the extenders and how they can be set to any length, as the situation requires. These strips are built of chains of hexagons lying between parallel lines. The strips can be traversed in one of two ways, as shown in Figure 2. The *sequential path* is used to simulate an edge that is used in a Hamilton circuit of $G$, and a *parallel* path for an unused edge. An example of how this works in $H$ is illustrated in figure 5.

The vertex gadgets are made of smaller components as listed below.

**extender** An extender is a 0 degree strip and has been described above.

**U-turn** A U-turn is used only in dark vertices. The U-turn is where a parallel path simulating an unused edge turns back on itself. As seen in Figure 3 (left) there are two distinct ways to traverse a U-turn, one is a continuous sequential path, and the other uses two parallel paths.

**rosette** A rosette is used to go from a horizontal strip to a strip of 60 or 120 degrees. The two ways to traverse a rosette, that is, sequential and parallel paths, are shown in Figure 3 (middle) .

**core** Every vertex has a core of three hexagons. The traversal of the core of a vertex gadget of degree

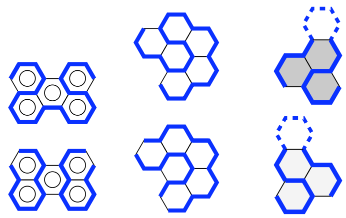

Figure 3: From left to right, U-turn, rosette, and core, with paths shown with bold lines.

three dictates which pair of edges are used in a Hamilton circuit and which are not. This is illustrated in Figure 3 (right). The cores of dark and light vertices are reflections of each other.

We combine the components to come up with six distinct types of vertex gadget. The entire collection of degree 3 vertex gadgets is shown in Figure 4. A degree 2 gadget is a subset of the shown gadgets omitting all hexagons beyond the core that are associated with the third connection to a strip.

It is not hard to show that we can assemble the individual gadgets to simulate an entire drawing of $G$. A portion of such an assembly is shown in Figure 5, and this ultimately becomes the instance of HCH, $H$.

We have done some calculations to give a polynomial upper bound for the space requirements of the drawing $H$. Our space requirements are a function of the space requirements of the rectilinear layout. Both the height and width of the rectilinear layout are $O(n)$ [5, 6]. It will be useful to use the constant $c_1 = 2\sin(60°)$, which represents the vertical height of each hexagon. We construct $H$ so that the height of every vertex gadget is exactly $16c_1$. We impose a minimal vertical distance between vertex gadgets of $2c_1$. Thus we have the height of $H$: $\text{height}(H) \leq 16c_1n + 2c_1(n-1) = 18c_1n - 2c_1 \in O(n)$.

For the width requirement of the drawing we can bound the width of any 60 or 120 degree edge by the width required by such an edge that spans the height of the entire drawing. Letting $c_2 = \cos(60°)$ we have the maximum width spanned by any given edge is $c_2\text{height}(H)$ and the width of $H$: $\text{width}(H) \leq nc_2\text{height}(H) \in O(n^2)$.

Thus the space requirements of $H$, $\text{height}(H) \times \text{width}(H) \in O(n^3)$. See Figure 5 for a drawing of the bottom part of $H$.

We summarize the current development by stating the following lemma, whose proof follows immediately from our construction.

**Lemma 2** *Transformation $T$ is a polynomial transformation from an instance of HCB to an instance of HCH.*

We now address the issue of Hamilton circuits in the constructed grid graph.

**Lemma 3** *Every Hamilton circuit in $G$ implies a Hamilton circuit in $H$.*

**Proof.** We have described how a Hamilton path in $G$ yields a Hamilton path in $H$. We traverse the gadgets in $H$ in such a way that simulates using the edges of the Hamilton circuit in $G$. □

**Lemma 4** *Every Hamilton circuit in $H$ implies a Hamilton circuit in $G$.*

**Proof.** Suppose we have a Hamilton circuit in $H$. One can verify that every strip can be traversed in exactly one of two ways, either by a single path, or by two parallel paths. We now focus on the part of the circuit that passes through each vertex gadget. Observe that all vertices, both dark and light of all degrees consist of components as shown in Figure 3 and extenders as shown in Figure 2. Each component forces one of two possible traversals, that is, sequential or parallel. Thus, given a Hamilton circuit in $H$, we obtain the obvious circuit in $G$ as described in the previous lemma. □

We have shown that HCH is polynomial reducible from HCB, proving Theorem 1, and as a result we conclude that HCH is NP-complete.

## References


[1] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman, 1979.

[2] A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM J. Comput.*, 11:676–686, 1982.

[3] D. S. Johnson and C. H. Papadimitriou. Computational complexity. In Lawler, Lenstra, Rinnooy Kan, and Shmoys, editors, *The Travelling Salesman Problem*, pages 37–85. Wiley, 1985.

[4] V. Polishchuk, E. Arkin, and J. Mitchell. Hamiltonian cycles in triangular grids. In *Proceedings of the 18th Canadian Conference on Computational Geometry*, pages 63–66. CCCG'06, 2006.

[5] P. Rosenstiehl and R. E. Tarjan. Rectilinear planar layouts and bipolar orientations of planar graphs. *Discrete Comput. Geom*, 1:343–353, 1986.

[6] R. Tamassia and I. G. Tollis. A unified approach to visibility representations of planar graphs. *Discrete Comput. Geom*, 1:321–341, 1986.

[7] C. Umans and W. Lenhart. Hamiltonian cycles in solid grid graphs. In *Proc. 38th Symp. FoCS*, pages 496–507, 1997.

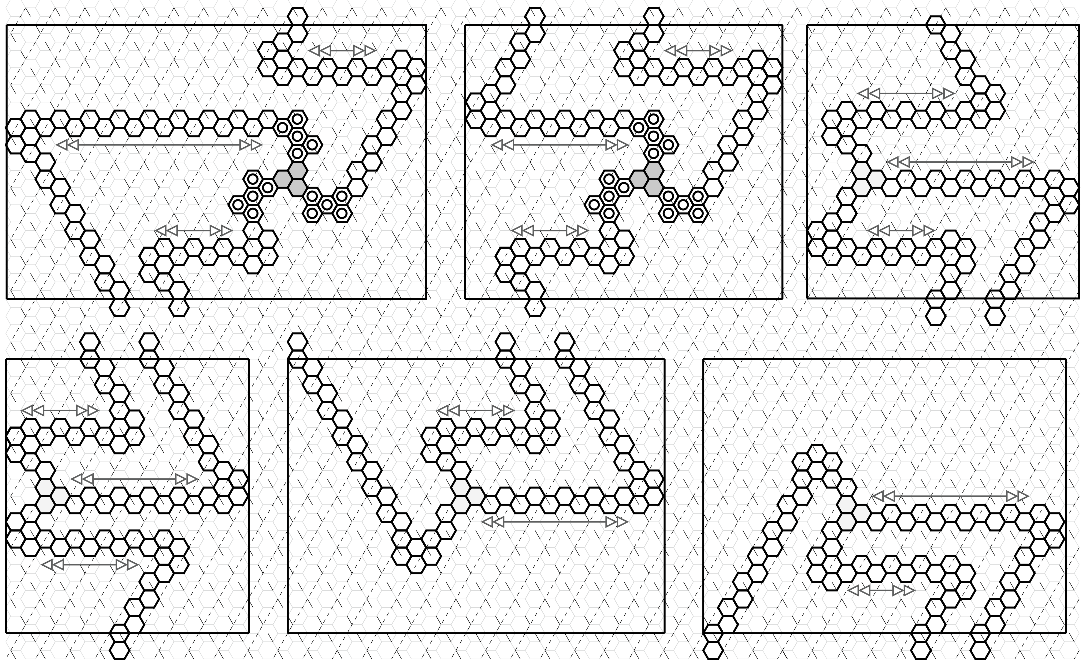

Figure 4: There are six distinct vertex gadgets. The anatomy of a vertex can be broken down to the components we call the core, a U-turn, an extender, and a rosette. These components are found in a bounding rectangle of the vertex gadget. The core of each vertex is shaded either dark or light. Each hexagon in a U-turn is decorated with a circle. At the rectangle boundary the vertex gadgets meet with strips that simulate edges. Note that the strips are constructed so that they all fit in a band between two parallel lines. Maintaining the strips between the bands ensure that strips connecting two vertex gadgets are aligned. Achieving alignment is done within the vertices.

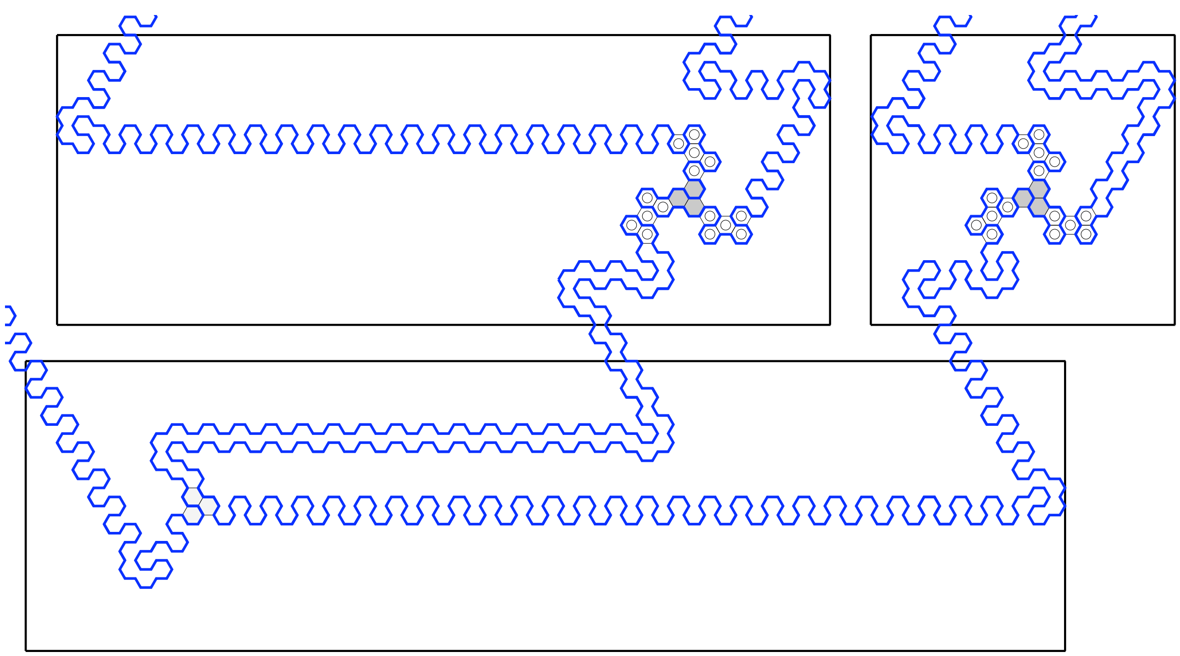

Figure 5: A portion of the graph H is shown highlighting a part of the Hamilton circuit in $G$ as shown in Figure 1. Consider the edges between the two dark vertices and the light vertex at the bottom of $G$. Only the edge on the right appears in the Hamilton circuit, so we see here how the sequential and parallel paths differ in $H$.