Dynamos in Three-Dimensional Meshes

Yoel de la Noval
Facultad de Matematica y Computation, Universidad de la Habana, Cuba

Fabrizio Luccio
Dipartimento di informatica, Università di Pisa, Italy

Yurai Nuñez Rodriguez
Facultad de Matematica y Computation, Universidad de la Habana, Cuba

Linda Pagli
Dipartimento di informatica, Università di Pisa, Italy
E-mail: yoel,yurai@comuh.uh.cu, luccio,pagli@di.unipi.it

Abstract

In a distributed system with dynamic behaviour based on majority rules, a *dynamo* is a pattern of initial faults which may lead the entire system to fail. The properties of dynamos have been extensively studied for different plane topologies, and for the butterfly family of connections. Here we investigate dynamos in three-dimensional toroidal meshes, as an approach to the study of majority-based fault tolerance in multi-dimensional structures. We establish lower and upper bounds on the number of faulty elements needed for a system break-down, both for irreversible and monotone failures, under two basic majority rules.

Key words: distributed computing, three-dimensional mesh, majority rule, dynamo, fault tolerance.

1. Introduction

In a distributed system, faulty elements can propagate wrong information through their neighbors. A formal tool to study fault propagation is *majority voting*, which applies when a vertex v performs an examination of the different copies of crucial data distributed among its neighbours [20]. If the majority of such neighbours has corrupted data, the data of v also become corrupted, and v is henceforth indistinguishable from a faulty vertex.

Majority voting is largely applied in distributed protocols for consensus, data base consistency, mutual exclusion, cryptographic key distribution, and other applications of distributed computing. The process that takes place in the network obeys, in essence, to the following elementary model. Initially each vertex is in one of two states (colors), *black* = *faulty* or *white* = *non-faulty*. Assuming that the network works synchronously, all vertices recolor themselves either black or white at each step, according to the color of the "majority" of their neighbors. Color propagation depends on how majority is defined. A major problem is to study the initial assignments of colors from which, after a finite number of steps, an all-black fixed point is reached. After Peleg [21], an initial set of black vertices causing such a total degradation is called *dynamo*, short for "dynamic monopoly". Note that this process may be asynchronous as well, as the local clocks may not be synchronized.

If each fault in the network is permanent, a dynamo is called *irreversible*. If a fault can instead be mended by majority, the dynamo is *reversible*, and a vertex may switch color several times according to a changing neighbourhood. In the latter case, the dynamo is *monotone* if the set of black vertices $B(t)$ existent at time t is a proper subset of $B(t+1)$, for all t. Clearly irreversible dynamos are always monotone.

The dynamics of majority rules have been extensively studied for cellular automata, and for certain families of graphs. Research has focused on the periodic dynamics of finite graphs [11, 23]; infinite graphs [18]; finite rings [2, 12]; and lines [16, 17]. Dynamic majority has also been applied to the study of the immune system [1], and to image processing [10]. Some results in distributed systems are related to catastrophic fault patterns [5, 19, 24], and "monopolies" (i.e., configurations that converge to the all black state in a single step) [3, 4, 20].

A general study of dynamos is quite recent. For the monotone case, some general lower and upper bounds on the size of dynamos have been estabilished in [21], and specific bounds for two-dimensional toroidal meshes of different types have been given in [9].The speed of convergence to a final configuration is studied in [7]. Irreversible dynamos have been examined for chordal rings [6], two-dimensional meshes [8], and butterflies [15].

In this paper we consider irreversible and monotone dynamos in three-dimensional toroidal meshes focusing on size, that is, on the minimum number of initial black elements needed to reach the fixed point. While meshes constitute one of the simplest and most natural ways of connecting processors in a network, not much has been done in more than two dimensions. In particular the study of dynamos for this case is new, while the problems arising are much harder to solve. We then consider our contribution as an approach to the study of majority-based fault tolerance in higher dimensions. Our results are summarized in table .

2. Basic Definitions

Let us consider an $m \times n \times p$ mesh M and denote with $v_{x,y,z}$, $0 \le x \le m-1$, $0 \le y \le n-1$, $0 \le z \le p-1$, a vertex of M. If M is closed as a torus, the vertices in each of the border planes are connected to the vertices in corresponding positions in the opposite border plane, thus forming ring connections in each of the three dimensions. Formally, we have:

Definition 1 Toroidal Mesh
A toroidal mesh of $m \times n \times p$ *vertices is a mesh where each vertex* $v_{x,y,z}$, $0 \le x \le m-1$, $0 \le y \le n-1$, $0 \le z \le p-1$, *is connected to the six vertices* $v_{(x-1) \bmod m,y,z}$, $v_{(x+1) \bmod m,y,z}$, $v_{x,(y-1) \bmod n,z}$, $v_{x,(y+1) \bmod n,z}$, $v_{x,y,(z-1) \bmod p}$, $v_{x,y,(z+1) \bmod p}$.

	Simple Majority		Strong Majority	
	Lower Bound	*Upper Bound*	*Lower Bound*	*Upper Bound*
Irreversible Dynamos	$mn/3 + \Theta(1)$	$(mn + mp + np)/3$ $+\Theta(m + n + p)$	$mnp/4 + \Theta(1)$	$mnp/4$ $+\Theta(mn + mp)$
Monotone Dynamos	$2mn/3 + \Theta(1)$	$2(mn + mp + np)/3$ $+\Theta(m + n + p)$	$2mnp/5 + \Theta(1)$	$2mnp/5$ $+\Theta(mn + mp)$

Table 1: Minimum cardinalities of irreversible and monotone dynamos for toroidal meshes of $m \times n \times p$ vertices. Θ denotes a strict asymptotic order of magnitude. When the formulae are not symmetric in m, n, p, apply the values yielding highest lower bounds, or lowest upper bounds. (For example in the lower bound $mn/3 + \Theta(1)$, m and n have the greatest values among m, n, p).

For covenience, toroidal meshes will be displayed in a Cartesian space x, y, z, with X, Y and Z denoting the three faces of the meshes lying in the coordinate planes orthogonal to the axes x, y and z, respectively. For example X includes the $n \times p$ vertices $v_{0,y,z}$ with $0 \leq y \leq n-1$, $0 \leq z \leq p-1$. Similarly, Y contains $m \times p$ vertices, and Z contains $m \times n$ vertices. X and Y share an edge containing the p vertices $v_{0,0,z}$ with $0 \leq z \leq p-1$, etc.

Majority is defined as follows [21]:

Definition 2 Irreversible-majority rule. *A vertex* v *becomes black if the majority of its neighbours are black. In case of tie* v *becomes black* (simple majority)*, or keeps its color* (strong majority). Reversible-majority rule. *A vertex* v *takes the color of the majority of its neighbours. In case of tie* v *becomes black* (simple majority)*, or keeps its color* (strong majority).

Note that black (i.e. faulty) vertices tend to propagate their faults. In the irreversible and reversible cases, simple (respectively: strong) majority asks for the presence of at least three (respectively: four) black neighbours to color black a vertex. In the reversible case, at least four white neighbours are needed to change the color from black to white. We can now formally define dynamos.

Definition 3 *A* simple *(respectively:* strong*)* irreversible dynamo *is an initial set of black vertices from which an all black configuration is reached in a finite number of steps under the simple (respectively: strong) irreversible-majority rule.*
A simple *(respectively:* strong*)* monotone dynamo *is an initial set of black vertices from which an all black configuration is reached in a finite number of steps under the simple (respectively:*

strong) reversible-majority rule, such that no black vertex ever turns white during the process (monotone black propagation).

Reversible majority is applied here only in the case of monotone dynamos, where it never happens that a black vertex has a majority of white neighbours. Then any set of black vertices reached during the process properly includes all the black sets reached at previous steps. Note that irreversible dynamos always exhibit a monotone black propagation.

Definition 4 *A k-*white *(respectively: k-*black*) *block *W is a restriction of the mesh to a subset of all white (respectively: all black) vertices, each of which has at least k neighbours in W.*

In particular a whole white plane section on the mesh orthogonal to one of the coordinate axes is a 4-white block (recall the torus closure at the borders). Two adjacent whole white rows form a 3-white block. A whole black three-dimensional rectangle R_S, of sizes $m_S \times n_S \times p_S$, with $m_S < m, n_S < n, p_S < p$, is a 3-black block (the corner vertices have three black neighbours).

The interest of detecting blocks in the mesh should be clear. A white block W is a set of white vertices that will never turn black under the chosen majority rule. The presence of W prevents the existence of a dynamo. A black block is a set of black vertices that will never turn white in the reversible case. In a monotone dynamo, all the black vertices must belong to 3-black blocks.

Finally, a new definition of tree introduced in [14] is crucially connected with the concept of block. We have:

Definition 5 [14] *A k-*dense tree *with $k \geq 1$ integer, is a graph $T = (V, E)$ with at least one vertex v (leaf) of degree $\leq k$, such that the restriction of T to $V - \{v\}$ has again at least one vertex of degree $\leq k$.*

Note that, for $k = 1$, the definition of k-dense tree coincides with the usual definition of tree. It has been shown in [14] that a k-dense tree of n vertices has at most $kn - k(k+1)/2$ edges.

3. Irreversible Dynamos

Network evolutions in irreversible and reversible dynamos are quite different. We treat the two cases separately, further dividing our discussion between simple and strong majority. Four cases then emerge, for each of which we derive upper and lower bounds to the size of dynamos.

3.1 Simple Irreversible Majority

Let us start with the construction of a dynamo following the simple irreversible majority rule. This gives an upper bound to the size of dynamos under such a rule.

Theorem 1 *An $m \times n \times p$ toroidal mesh M admits a simple irreversible dynamo S with $|S| = (mn + mp + np)/3 + \Theta(m + n + p)$.*

Proof (Constructive). Consider the three faces X, Y, Z of M. Recall that X and Y contain $n \times p$ and $m \times p$ vertices, respectively, and share a side e of p vertices. Refer to figure 1. Divide X in groups of three consecutive rows starting from the side opposite to e, and color the vertices of each group as follows. In the first row all vertices are white, except for the leftmost one; the second and the third row hold alternating colors, respectively starting with a black and with a white vertex. If n is not multiple of 3, make the same configuration without the last row, or the last two rows. This colors X up to side e. Continue on Y with the same coloring, in the $m-1$ rows following e. We have $\lceil \frac{n+m-1}{3} \rceil (p + 1)$ black vertices on the combination of faces $X|Y$. Color now Z as indicated in figure 1 (note that the leftmost column, and two bottom rows, are all white). We have $\lceil \frac{m-2}{3} \rceil (n - 1)$ black vertices on Z. In total we have $(mn + mp + np)/3 + \Theta(m + n + p)$.

In each group of three consecutive rows in $X|Y$, two of them become immediately black because each white vertex has three black neighbours. Also the leftmost column becomes black in the first step. In each consecutive step two new columns, adjacent to the ones colored black in the previous step, also become black until $X|Y$ is completely black. This makes black the two sides of X and Y common with Z (say, the leftmost column and the bottom row of Z in figure 1), so that Z is wholly colored black with a similar mechanism. Once the three orthogonal plane sections X, Y, Z are all black, the remaining $(m - 1) \times (n - 1) \times (p - 1)$ submesh becomes black starting from the eight corner vertices that have now three black neighbours in X, Y, Z. □

Recall now that the presence of a 4-white block in the mesh prevents any simple dynamo to exist. The following lower bound theorem crucially relies on this fact.

Theorem 2 *Let S be a simple irreversible dynamo for an $m \times n \times p$ toroidal mesh M. Then $|S| \geq max\ (mn/3, mp/3, np/3) + \Theta(1)$.*

Proof Consider the $n \times p$ chains of m vertices incident to the face X of M and parallel to the x axis. Some of them must include black vertices in order to prevent the existence of 4-white blocks. Project such black vertices onto X, along the direction x. Now the existence of a cycle (closed path) of white vertices in X clearly corresponds to a 4-white block in three dimensions, in the form of a cylinder parallel to x. (For example no white cycle exists on X with the pattern of black vertices shown in figure 1, then no cylindrical 4-white block parallel to x exists in that mesh). From a theorem for strong majority in two-dimensional meshes proved in [8], at least $\lceil \frac{np+1}{3} \rceil$ black vertices must exist in X to prevent the formation of white cycles. then at least the same number of black vertices must exist in M, to prevent, by their projections, the existence of cycles in X. Applying this argument also to the faces Y and Z the thesis immediately follows. □

3.2 Strong Irreversible Majority

Strong irreversible majority is simpler to study than the previous case. The following theorem establishes an upper bound to the size of dynamos.

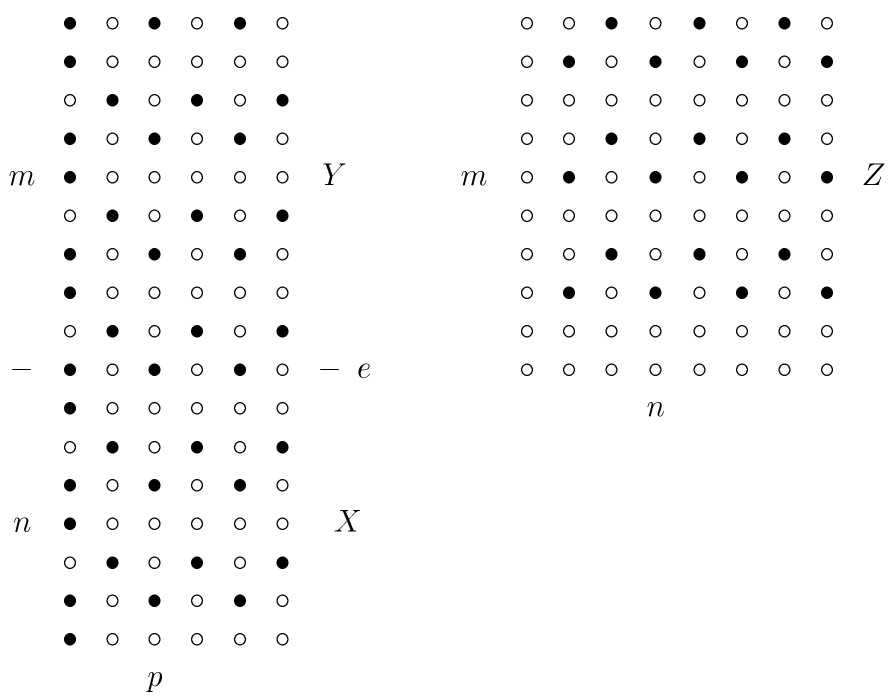

Figure 1: Node coloring for simple irreversible majority. In the example $m = 10$, $n = 8$, $p = 6$.

Theorem 3 *An $m \times n \times p$ toroidal mesh M admits a strong irreversible dynamo S with $|S| = mnp/4 + \Theta(mn + mp)$.*

Proof Consider the plane sections P_i of M, $0 \leq i \leq m-1$, parallel to face X and placed at integer x coordinates (in particular, $P_0 = X$). On the sections P_i, with i even, place black and white vertices as in a checkboard (figure 2), for a total of $\lceil \frac{np}{2} \rceil \cdot \lceil \frac{m}{2} \rceil$ black vertices. On the sections P_i, with i odd, place alternate black and white vertices on the two sides a, b laying on Y and Z (figure 2), for a total of $\lceil \frac{n+p-1}{2} \rceil \cdot \lfloor \frac{m}{2} \rfloor$ black vertices. The total number of black vertices is then $mnp/4 + \Theta(mn + mp)$.

The white vertices of all P_i, i even, become black in one step, because all of them have four black neighbours. Then, the white vertices of a and b on P_i, i odd, become black. Then all the other vertices of these sections become black starting from the four corner vertices v_1, v_2, v_3, v_4. $\square$

To determine a lower bound that matches the upper bound of theorem 3 we will make crucial use of the concept of k-dense tree. In fact, to change the color of all the white vertices into black, at least one of them must have at most two white neighbours at any step. That is, the restriction of the mesh to the subgraph of white vertices must be a 2-dense tree. We have:

Theorem 4 *Let S be a strong irreversible dynamo for an $m \times n \times p$ toroidal mesh M. Then $|S| \geq mnp/4 + \Theta(1)$.*

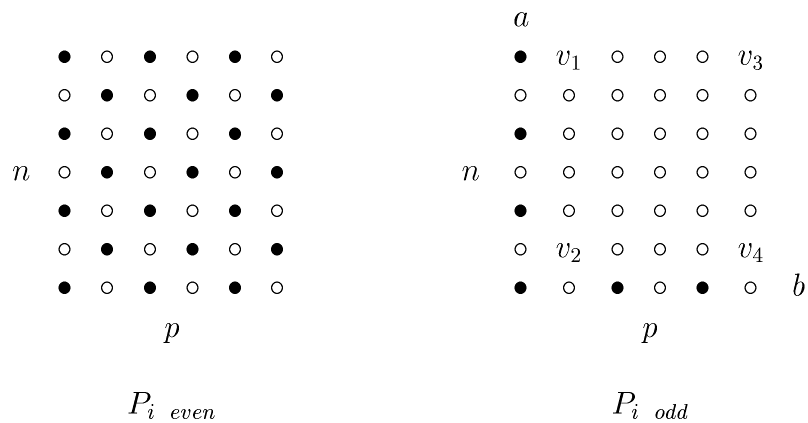

Figure 2: Node coloring for strong irreversible majority.

Proof Let T and E be the sets of vertices and edges of the mesh. Let B and $W = T - B$ be the subsets of black and white vertices, respectively. And let E_W be the subset of edges connecting pairs of white vertices. Consider the subset $E - E_W$ of edges having at least one black extreme. We have $|E| - |E_W| \leq 6|B|$, where the equality is met when each vertex in B is connected with exactly one vertex in W. As already observed, white vertices must form a 2-dense tree, that is $|E_W| \leq 2|W| - 2 \cdot 3/2 = 2|T| - 2|B| - 3$ (see section 2 for the upper bound on the number of edged in a k-dense tree). Combining the two inequalities above we have: $|E| \leq |E_W| + 6|B| \leq 2|T| + 4|B| - 3$. Noting that $|T| = mnp$ and $|E| = 3mnp$, we immediately derive: $|B| \geq mnp/4 + 3/4$. $\square$

4. Monotone Dynamos

As for the irreversible case, color propagation crucially depends on the type of majority. Here, however, we must "protect" black vertices to prevent them from becoming white.

4.1 Simple Monotone Majority

An upper bound to the size of dynamos with simple monotone majority is given in the following theorem. The novelty, compared with the irreversible case, is that black vertices must now be clustered into 3-black blocks to guarantee monotonicity.

Theorem 5 *An $m \times n \times p$ toroidal mesh M admits a simple monotone dynamo S with $|S| = 2(mn + mp + np)/3 + \Theta(m + n + p)$*

Proof As in theorem 1, consider the faces X, Y of M with the common side e. Refer to figure 3. Color X and Y by setting two black columns for each group of three, and two black rows on X at the side opposite to e. We then have $\lceil 2p/3 \rceil (m + n - 1) + \lfloor 2p/3 \rfloor$ black vertices on the combination of faces $X|Y$. Assume that the leftmost colums of $X|Y$

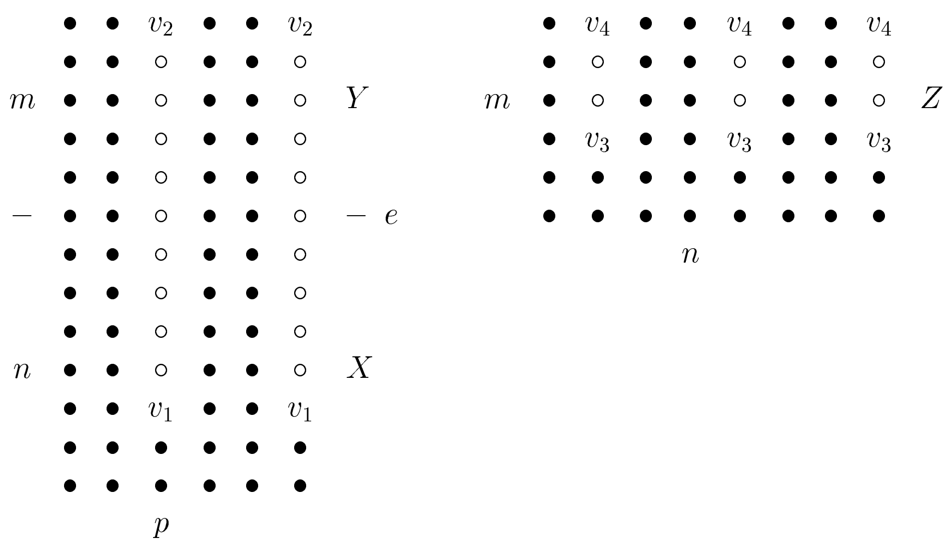

Figure 3: Node coloring for simple monotone majority.

lie on the two sides of X and Y common to Z (say, the left column and the bottom row of Z). These two sides then appear black in Z. Color the remaining vertices of Z as indicated. We have $\lceil 2(n-2)/3 \rceil (m-1) + \lceil (n-1)/3 \rceil$ new black vertices in Z, for a total of $2(mn+mp+np)/3 + \Theta(m+n+p)$ black vertices in M.

All the white vertices of $X|Y$ become black in pairs, starting from the pairs marked v_1, v_2 that have three black neighbours from the beginning. Similarly, the white vertices of Z become black starting from the pairs marked v_3, v_4. Once the three faces X, Y, Z are all black, the whole mesh is colored black as indicated in the proof of theorem 1. To complete the proof note that all the initial black vertices form a 3-black block, and each new black vertex is adjoined to such a block, so that no black vertex ever turns white. □

To derive a lower bound to the size of dynamos, recall that the presence of 4-white blocks prevents a dynamo to exist. We have:

Theorem 6 *Let S be a simple monotone dynamo for an $m \times n \times p$ toroidal mesh M. Then $|S| \geq max\ (2np/3, 2mp/3, 2mn/3) + \Theta(1)$.*

Proof As in theorem 2, project the black vertices of M onto X along the direction x. On this new configuration, let T and E be the sets of vertices and edges of X; B be the subset of black vertices in X; E_W be the subset of edges in X with two white extremes. For S to be a dynamo of M, no white cycle must be present in X, because one such a cycle would corrispond to a 4-white block in M in the form of a cylinder parallel to x. That is, the white vertices in X must form a forest. We then have: $|E_W| \leq |T| - |B| - 1$, where $T - B$ is the subset of white vertices in X.

Since the black vertices of M must form a 3-black block, and each vertex has only two neighbours with the same y and z coordinates, each black vertex $v \in B$ must be adjacent to at least another black vertex $u \in B$. In fact, B can be partitioned into three subsets B_1, B_2, B_3 (figure 4) such that B_1 is a set of 1-dense trees (or simply a forest) with at least

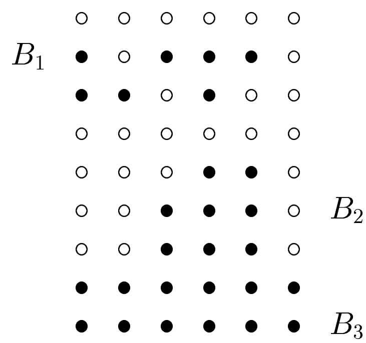

Figure 4: Projections on X of 3-black blocks.

two vertices each; B_2 is a set of 2-dense trees with all vertices of degree ≥ 2; and B_3 is a set of 3-dense trees with all vertices of degree ≥ 3. In particular, each dense tree in B_3 must occupy two whole adjacent rows or columns. Consider now the set $E - E_W$ of edges in X having at least one black extreme. We have: $|E| - |E_W| \leq \frac{7}{2}|B_1| + \frac{12}{4}|B_2| + \frac{5}{2}|B_3|$, where the equality is met when each set of B_1 contains exactly two elements, with a total of seven incident edges in X; each set of B_2 contains exactly four elements, with a total of twelve incident edges in X; and each set of B_3 is composed of two adjacent rows, or two adjacent columns, so that each of its elements has one edge connected to a white vertex and three edges connected to black vertices, with an average of $5/2$ incident edges per vertex in X.

Combining the two inequalities derived thus far; noting that $|E| = 2np$ and $|T| = np$; and letting $|B_1| = \alpha_1|B|$, $|B_2| = \alpha_2|B|$, $|B_3| = (1 - \alpha_1 - \alpha_2)|B|$ we have: $|E| \leq |T| - |B| - 1 + (\frac{7}{2}\alpha_1 + 3\alpha_2 + \frac{5}{2}(1 - \alpha_1 - \alpha_2))|B|$, hence $|B| \geq \frac{2(np+1)}{2\alpha_1+\alpha_2+3}$. Note now that for a black vertex $v \in B_1$ to be part of a 3-black block, all the m vertices of M with the same y and z coordinates of v must be black. Similarly, for each black vertex $v \in B_2$, at least another vertex of M with the same y and z coordinates of v must be black, while the vertices in B_3 already form a 3-black block. Letting B_M be the set of black vertices in M we then have: $|B_M| \geq m|B_1| + 2|B_2| + |B_3| \geq \frac{2(np+1)}{2\alpha_1+\alpha_2+3}(\alpha_1(m-1) + \alpha_2 + 1)$. It can be immediately verified that this expression, with the obvious condition $m \geq 3$, is minimized for $\alpha_1 = \alpha_2 = 0$ (i.e., all the black vertices in X belong to B_3). Hence we have: $|B_M| \geq \frac{2(np+1)}{3}$.

Applying the same argument to the projections of black vertices onto Y and Z, the thesis immediately follows. $\square$

4.2 Strong Monotone Majority

In this final subsection we impose monotonicity under strong majority, with a combination of the arguments used for strong irreversible majority (subsection 3.2), and simple monotone majority (subsetioon 4.1). In particular, black vertices must be clustered into 3-black blocks. As before, the upper bound on the size of dynamos is constructive. We have:

Theorem 7 *An $m \times n \times p$ toroidal mesh M admits a simple monotone dynamo S with $|S| = 2mnp/5 + \Theta(mn + mp)$*

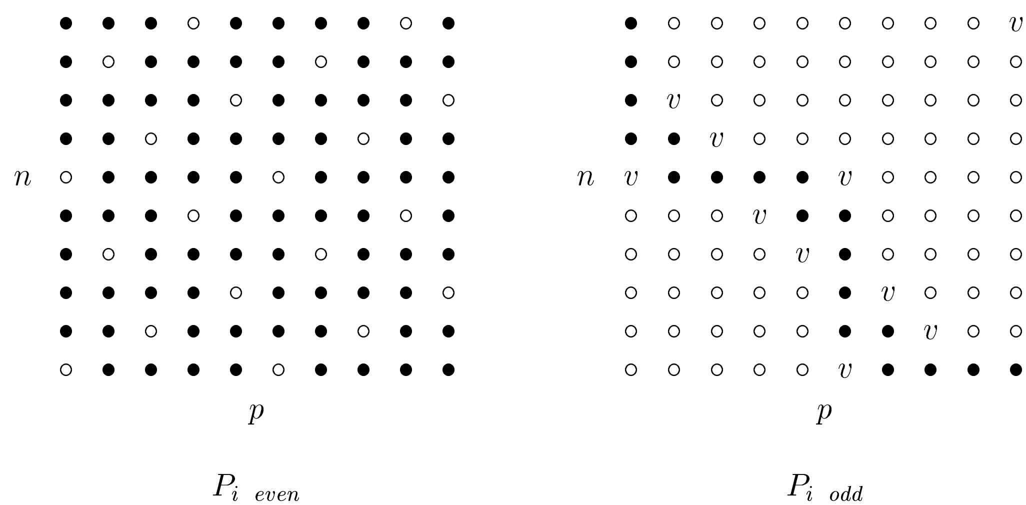

Figure 5: Node coloring for strong monotone majority.

Proof As in the proof of theorem 3, consider the plane sections P_i of M, $0 \leq i \leq m-1$, parallel to the face X, with $P_0 = X$. Refer to figure 5. On each of the sections P_i, with i even, place the white vertices at "knight's distance", such that each white vertex has four black neighbours, and each black vertex has three black neighbours. Note that there are five white vertices in each square portion of P_i of side five, hence there are $\lceil \frac{20}{25} np \rceil$ black vertices in each section P_i, with i even, for a total of $\lceil \frac{20}{25} np \rceil \lceil \frac{m}{2} \rceil \geq 2mnp/5$ black vertices in all such sections. On the sections P_i, with i odd, place the black vertices along a path from the top-left corner to the bottom-right corner, such that the vertices in the same positions in the sections P_i of even indices are all black, for a total of $(n+p-1)\lfloor \frac{m}{2} \rfloor$ black vertices. These black vertices have one or two black neighbours in P_i, plus two black neighbours in P_{i-1} and P_{i+1}. We then have a total of $2mnp/5 + \Theta(mn + mp)$ black vertices in M, all clustered in a 3-black block.

The white vertices of all P_i, i even, become black in one step, because all of them have four black neighbours. Then, the white vertices on P_i, i odd, become black in successive steps, starting from the vertices marked v in the figure. $\square$

To prove a matching lower bound, recall that to change the color of all white vertices into black, the restriction of the mesh to such vertices must be a 2-dense tree. We have:

Theorem 8 *Let S be a simple monotone dynamo for an $m \times n \times p$ toroidal mesh M. Then $|S| \geq 2mnp/5 + \Theta(1)$.*

Proof As in the proof of theorem 4, let T and E be the sets of vertices and edges of the mesh; B and $W = T - B$ be the subsets of black and white vertices. Furthermore, let E_{WW} and E_{WB} be the subsets of edges connecting two white vertices, and a white and a black vertex, respectively. Since the white vertices must form a 2-dense tree we have: $|E_{WW}| \leq 2|T-B|-3$ (see section 2). Since the black vertices must be clustered into 3-black blocks, each black vertex can have at most three white neighbours, that is: $|E_{WB}| \leq 3|B|$. A relation between

$|E_{WW}|$ and $|E_{WB}|$ can be established by noting that each white vertex v has six incident edges, each one connecting v with a black vertex, or with another white vertex. That is: $|E_{WB}|+2|E_{WW}| = 6|T-B|$. Combining these relations we have: $2|T-B|+6 \leq |E_{WB}| \leq 3|B|$; and recalling that $|T| = mnp$ we finally obtain: $|B| \geq 2mnp/5 + 6/5$. □

5. Concluding Remarks

In this paper we have discussed the propagation of faulty behaviour in a distributed system, under various disciplines of majority voting among neighbours. In particular we have studied upper and lower bounds to the size of the configurations of faulty vertices (dynamos) that cause the whole system to fail. The upper bounds are constructive, that is, we have exhibited the dynamos of minimum size that we were able to derive. The lower bounds have instead been derived with combinatorial arguments, making use of the properties of a particular family of graphs called dense trees.

This classical problem had been previously studied for two-dimensional network topologies only. Here we have examined three-dimensional meshes with toroidal closures. The problems thus arising are much harder than in two dimensions. The proposed solutions are not always strict, in the sense that the upper and lower bounds to the size of dynamos agree in order of magnitude, but have identical terms of higher order only in the case of strong majority. See table for a summary of results. Finding bounds matching exactly, as done in the literature for two-dimensional meshes, is an open problem of not easy solution.

We regard this work as an approach to the study of majority-based fault tolerance in higher dimensions. Several changes in the topology are obviously possible. In particular, our discussion has been directed to meshes with toroidal connections, to avoid the border effects occurring in simple meshes. Our techniques, however, can be immediatly extended to this case.

References

[1] Z. Agur. Fixed points of majority rule cellular automata with application to plasticity and precision of the immune system. *Complex Systems*, 2:351–357, 1988.

[2] Z. Agur, A. S. Fraenkel, and S. T. Klein. The number of fixed points of the majority rule. *Discrete Mathematics*, 70:295–302, 1988.

[3] J-C Bermond, D. Peleg. The power of small coalitions in graphs. In *Proc. of 2nd Colloquium on Structural Information and Communication Complexity*, 173-184, 1995.

[4] J-C Bermond, J. Bond, D. Peleg, S. Perennes. Tight bounds on the size of 2-monopolies In *Proc. of 3rd Colloquium on Structural Information and Communication Complexity*, 170-179, 1996.

[5] R. De Prisco, A. Monti, L. Pagli. Efficient testing and reconfiguration of VLSI linear arrays. To appear in *Theoretical Computer Science*.

[6] P. Flocchini, F. Geurts, N. Santoro. Dynamic majority in general graphs and chordal rings. In *Proc. 25-th Intern. Workshop on Graph Theoretic Concepts in Computer Science*, Ascona 1999.

[7] P. Flocchini, R. Kralovic, A. Roncato, P. Ruzicka, N. Santoro. On time versus size for monotone dynamic monopolies in regular topologies. *Proc. of 7-th Colloquium on Structural Information and Communication Complexity*, L'Aquila 2000 (to appear).

[8] P. Flocchini, E. Lodi, F. Luccio , L. Pagli, and N. Santoro. Irreversible dynamos in tori. In *Proc. EUROPAR 98* Southampton 1998, 554–562.

[9] P. Flocchini, E. Lodi, F. Luccio , L. Pagli, and N. Santoro. Monotone dynamos in tori. In *Proc. of 6-th Colloquium on Structural Information and Communication Complexity*, Lacanau 1999, 152-165.

[10] E. Goles and S. Martinez. Neural and automata networks, dynamical behavior and applications. *Maths and Applications*, Kluwer Academic Publishers, 1990.

[11] E. Goles and J. Olivos. Periodic behavior of generalized threshold functions. *Discrete Mathematics*, 30, 1980, 187–189.

[12] A. Granville. On a paper by Agur, Fraenkel and Klein. *Discrete Mathematics*, 94, 1991, 147–151.

[13] N. Linial, D. Peleg, Y. Rabinovich and M. Sachs. Sphere packing and local majority in graphs In *Proc. of 2nd ISTCS*, IEEE Comp. Soc. Press. 1993, 141-149.

[14] F. Luccio and L. Pagli. Dense Trees: a new structure forinterconnection. In *Proc. Distributed Data and Structures 2*, Princeton 1999, 56–72.

[15] F. Luccio, L. Pagli, and H. Sanossian. Irreversible dynamos in butterflies. In *Proc. of 6-th Colloquium on Structural Information and Communication Complexity*, Lacanau 1999, 204-217.

[16] G. Moran. Parametrization for stationary patterns of the r-majority operators on 0–1 sequences. *Discrete Mathematics*, 132, 1994, 175–195.

[17] G. Moran. The r-majority vote action on 0–1 sequences. *Discrete Mathematics*, 132, 1994, 145–174.

[18] G. Moran. On the period-two-property of the majority operator in infinite graphs. *Transactions of the American Mathematical Society*, 347(5):1649–1667, 1995.

[19] A. Nayak, N. Santoro, and R. Tan. Fault intolerance of reconfigurable systolic arrays. *Proc. 20th International Symposium on Fault-Tolerant Computing*, 202-209, 1990.

[20] D. Peleg. Local majority voting, small coalitions and controlling monopolies in graphs: A review. In *Proc. of 3rd Colloquium on Structural Information and Communication Complexity*, Siena 1996, 152-169.

[21] D. Peleg. Size bounds for dynamic monopolies. In *Proc. of 3rd Colloquium on Structural Information and Communication Complexity*, Ascona 1997, 151-161.

[22] S. Poljak. Transformations on graphs and convexity. *Complex Systems*, 1:1021–1033, 1987.

[23] S. Poljak and M. Sůra. On periodical behavior in societies with symmetric influences. *Combinatorica*, 1:119–121, 1983.

[24] N. Santoro, J. Ren, A. Nayak. On the complexity of testing for catastrophic faults. In *Proc. of 6th International Symposium on Algorithms and Computation*, 188-197, 1995.