Sums of Uncertainty: Refinements go gradual

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POPL 2017

Gradual typing

I'm last in the session, so I'll keep this brief.

Traditionally, gradual typing is about

- migrating **incrementally** (gradually)
  from dynamically typed code
  to statically typed code.

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. . . I lost her at "dynamically typed".

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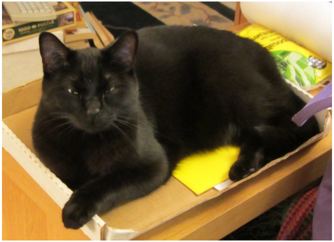

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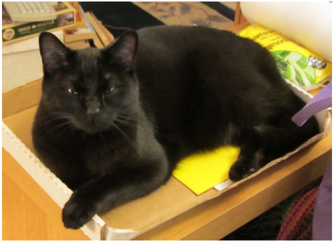

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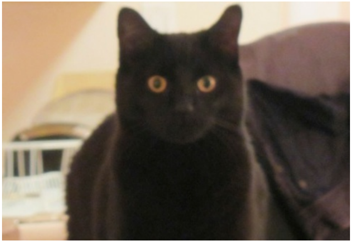

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Our paper has (a simplified form of) what were once called refinement types, which we now call datasort refinements.

Standard ML: dynamically typed?

$$\mathsf{datatype\ nat} \ = \ \mathtt{Zero} \mid \mathtt{Succ}\ \mathsf{of\ nat}$$

$$\begin{array}{l} \mathsf{case}\ x : \mathsf{nat}\ \mathsf{of} \\ \quad \mathtt{Zero} \Rightarrow \mathtt{...} \\ \ \mid \mathtt{Succ}\ y \Rightarrow \mathtt{...} \end{array}$$

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But the Definition requires compilers to accept **nonexhaustive** matches:

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$$\begin{aligned} &\texttt{case}\ x : \texttt{nat}\ \texttt{of} \\ &\quad \texttt{Succ}\ y \Rightarrow \ldots \end{aligned}$$

If $x = \texttt{Zero}$, then the exception Match is raised.

This nonexhaustive match is fine,
if we know that x will never be Zero.

Refined Standard ML

Datasort refinements [Freeman & Pfenning 1991, Davies 2005, . . .]
push the knowledge that x is not `Zero` into the type system.

```
case x : nonzero of
  Succ y ⇒ ...
```

This **is** exhaustive, because x has **datasort** nonzero.

Datasorts

Datasorts refine ML datatypes

`datatype nat = Zero | Succ of nat`

- sum type : `Succ` **or** `Zero`
- recursive type: `datatype nat = Zero | Succ of nat`

Datasorts

Datasorts refine ML datatypes

datatype nat $=$ Zero | Succ of nat

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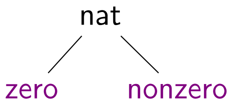
datasort zero $=$ Zero

datasort nonzero $=$ Succ of nat



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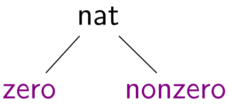

This paper: gradual, refined **sum types**.

Static sums

The usual type-theoretic sum type:

$\texttt{datatype}\ A_1 + A_2 \ =$
$\quad \texttt{inj}_1\ \texttt{of}\ A_1 \mid \texttt{inj}_2\ \texttt{of}\ A_2$

Elimination form: two-armed $\texttt{case}(e, \texttt{inj}_1\, x_1.e_1, \texttt{inj}_2\, x_2.e_2)$

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Subscript sums $A_1 +_1 A_2$ and $A_1 +_2 A_2$,
corresponding to datasort refinements:

$\mathtt{datasort}\ A_1 +_1 A_2 \ = \ \mathsf{inj}_1\ \mathsf{of}\ A_1$

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Elimination form: **one**-armed $\mathsf{case}(e, \mathsf{inj}_k\, x_k.e_k)$.

$$x : (\mathsf{Int} +_1 \mathsf{Bool}) \vdash \mathsf{case}(x, \mathsf{inj}_1\, x_1.x_1) : \mathsf{Int}$$

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Case expressions over $+$, $+_1$, $+_2$ **never** raise Match exceptions.

Dynamic sum

The **dynamic** sum type, corresponding to Standard ML:

$\texttt{datatype}\ A_1 +^? A_2\ =$
$\quad \texttt{inj}_1\ \texttt{of}\ A_1 \mid \texttt{inj}_2\ \texttt{of}\ A_2$

$+^?$ allows two-armed $\texttt{case}(e, \texttt{inj}_1\, x_1.e_1, \texttt{inj}_2\, x_2.e_2)$.

But $+^?$ **also** allows one-armed $\texttt{case}(e, \texttt{inj}_k\, x_k.e_k)$,
which may raise a $\texttt{Match}$ exception.

Gradual sums

					match failures are…
Standard ML				$+^?$	possible
refined SML	$+$	$+_1$	$+_2$		impossible

Gradual sums

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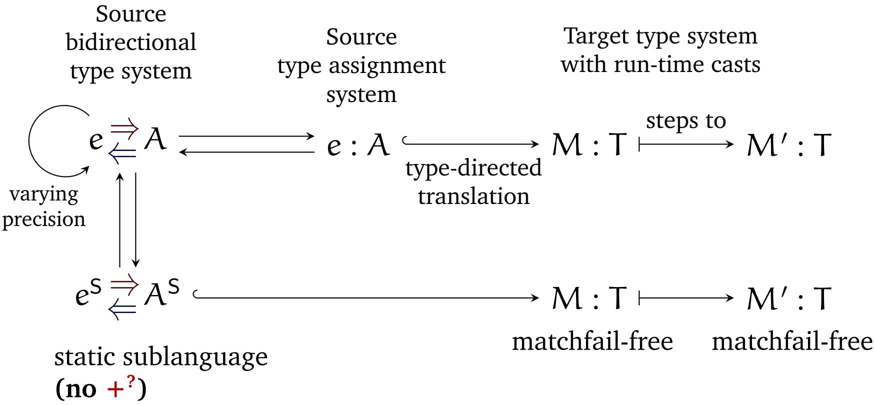
	Standard ML				$+^?$	possible
+	refined SML	$+$	$+_1$	$+_2$		impossible
=	**Gradual sums**	$+$	$+_1$	$+_2$	$+^?$	**possible** iff $+^?$ used in annotations

Road map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Source type assignment system
Target type system with run-time casts
$e \Rightarrow A$ / $e \Leftarrow A$
$e : A$
$M : T$
steps to
$M' : T$
type-directed translation
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static sublanguage
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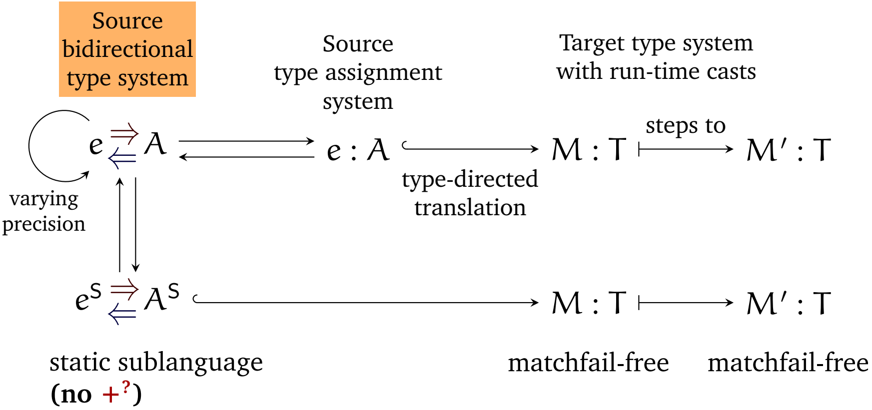

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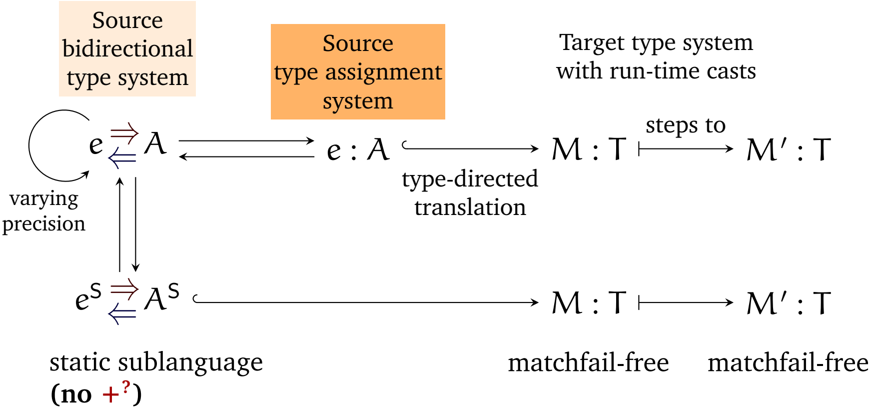

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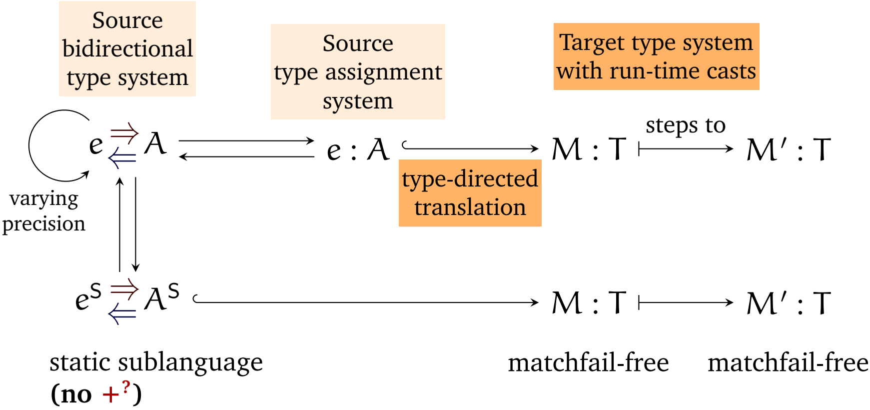
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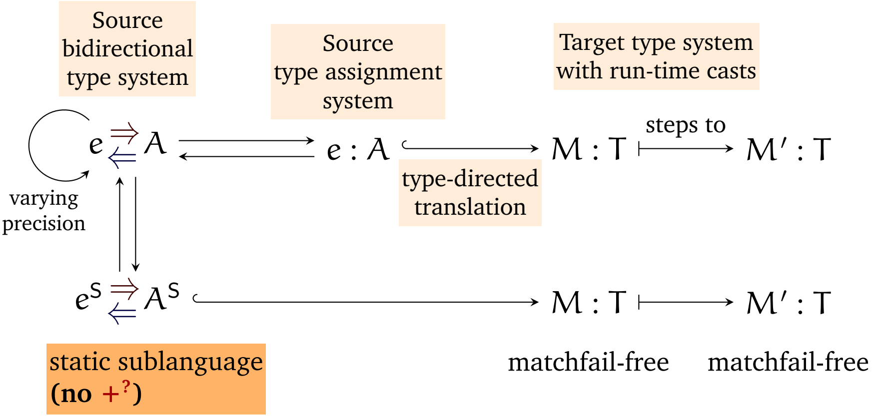

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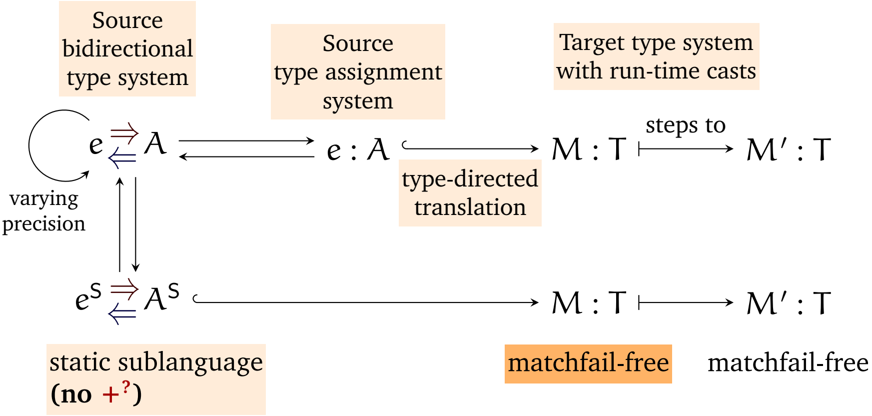

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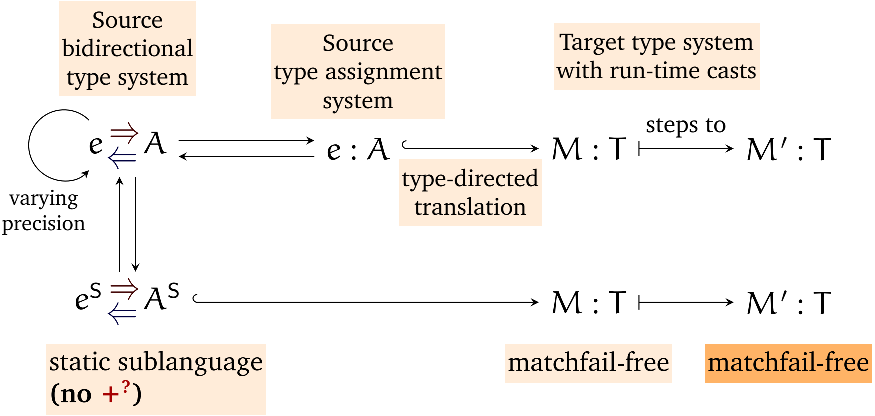
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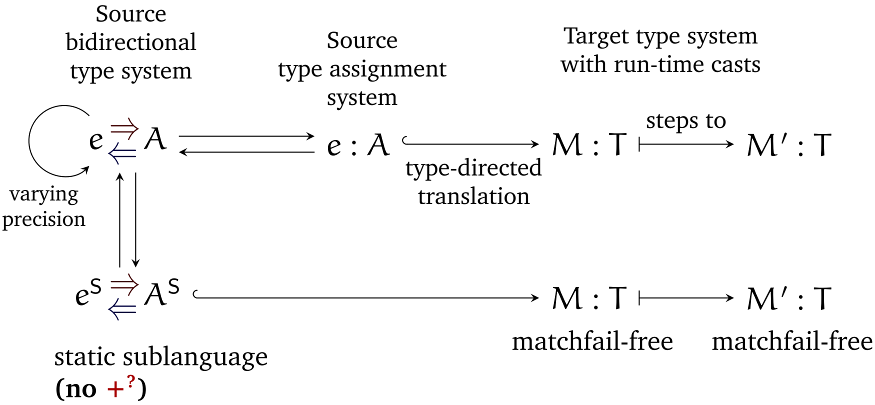

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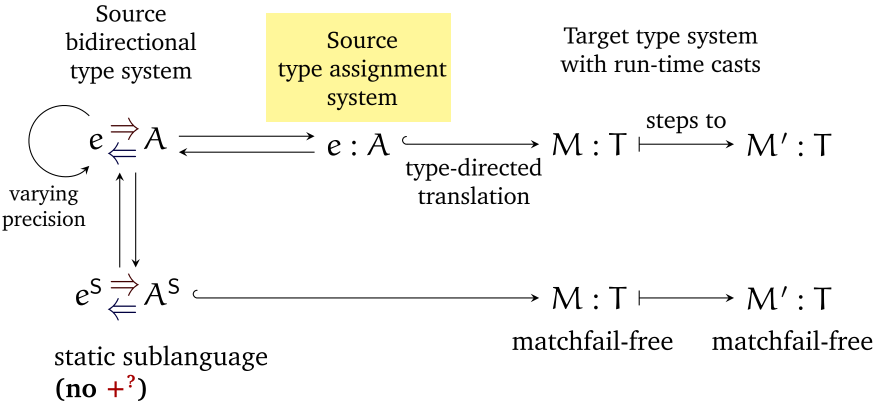

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Source type assignment

Design **introduction** and **elimination** rules:

- How are the static sums $+$, $+_1$, $+_2$ introduced and eliminated?
- How is the dynamic sum $+^?$ introduced and eliminated?

Static sums

Design **introduction** and **elimination** rules for $+_1$, $+_2$:

$$\frac{\Gamma \vdash e : A_k}{\Gamma \vdash (\mathtt{inj}_k\, e) : (A_1 +_k A_2)}\ {+_k}\text{Intro}$$

$$\frac{\Gamma \vdash e : (A_1 +_k A_2) \qquad \Gamma, x_k : A_k \vdash e_k : B}{\Gamma \vdash \mathtt{case}(e, \mathtt{inj}_k\, x_k.e_k) : B}\ {+_k}\text{Elim}$$

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Introduction rule for $+$ via subtyping:
$(\mathsf{inj}_k\, e) : (A_1 + A_2)$ because $(A_1 +_k A_2) \leq (A_1 + A_2)$.

$$\frac{\Gamma \vdash e : (A_1 + A_2) \qquad \begin{array}{c}\Gamma, x_1 : A_1 \vdash e_1 : B \\ \Gamma, x_2 : A_2 \vdash e_2 : B\end{array}}{\Gamma \vdash \mathsf{case}(e, \mathsf{inj}_1\, x_1.e_1, \mathsf{inj}_2\, x_2.e_2) : B}\ +\text{Elim}$$

(two-armed elimination for $+_k$ possible via subtyping)

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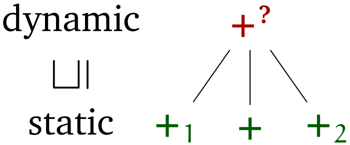
Replacing an annotation $A +_1 B$ with $A +^? B$ preserves typing (varying precision—gradual guarantee)

Replacing an annotation $A +^? B$ with a **more precise** annotation does not always preserve typing.

Defining precision

First, $\sqsubseteq$ on **sum constructors**

$+^?$, $+$, $+_1$, $+_2$:



Defining precision

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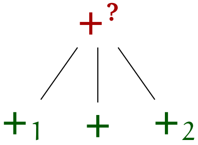
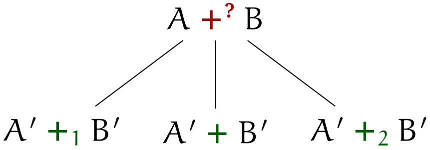
dynamic

$\sqsupseteq$

static



Extend $\sqsubseteq$ pointwise:
if $A' \sqsubseteq A$ and $B' \sqsubseteq B$ then...



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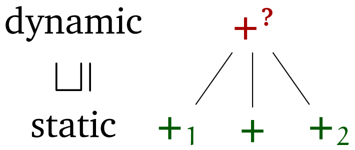

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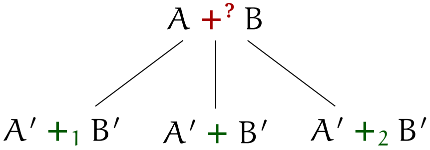
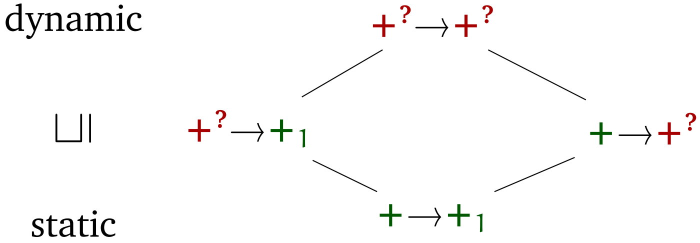


Other constructors **covariant** (similar to $\sqsubset$ in refinement types):



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- These 3 premises = **directed consistency** $A' \rightsquigarrow B'$

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$$\begin{array}{ccc} A' & & B' \\ \sqsupseteq & & \sqsupseteq \\ A & \leq & B \end{array}$$

Subsumption

- The usual subsumption rule:

$$\frac{\Gamma \vdash e : A \qquad A \leq B}{\Gamma \vdash e : B}$$

- In a land of imprecision: “kinda A”, “kinda B”

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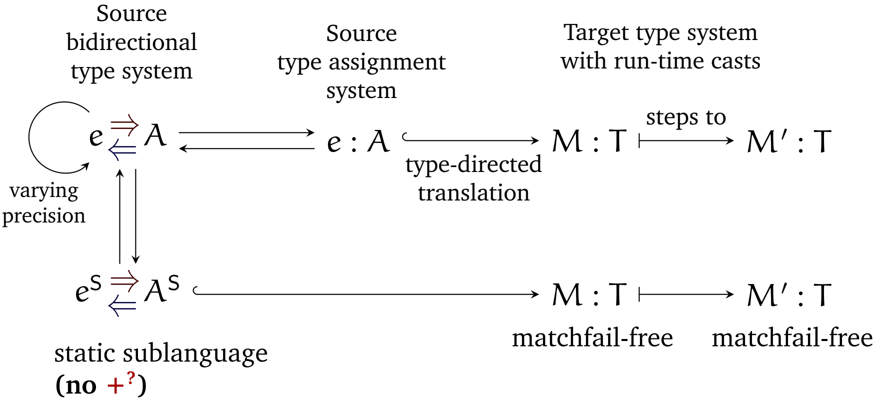
- Is directed consistency transitive?

Road map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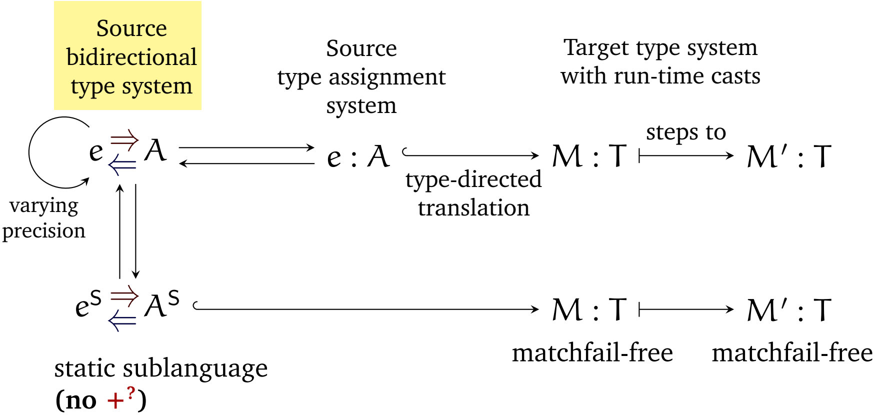

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Bidirectional typing: why?

Some past answers:

- to handle features beyond Damas–Milner
  (Pierce & Turner 2000;
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  Dunfield & Krishnaswami 2013; . . .)
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Here:

- to make typing **more predictable**,
  by avoiding **unnecessary imprecision**.

Bidirectional typing in one slide

- **Organize** the flow of information from type annotations:
  - Given Γ, e, and a known type A,
    check e:
    $$\Gamma \vdash e \Leftarrow A$$
  - Given Γ and e,
    synthesize a type for e:
    $$\Gamma \vdash e \Rightarrow A$$
- The type A in the checking judgment $e \Leftarrow A$ is a **goal**.

Bidirectional typing

Frank Pfenning's recipe:
intro rules check, elim rules synthesize.

$$\frac{\Gamma, x : A_1 \vdash e \Leftarrow A_2}{\Gamma \vdash \lambda x.\, e \Leftarrow A_1 \rightarrow A_2}\ \textsf{Chk}{\rightarrow}\textsf{Intro}$$

$$\frac{\Gamma \vdash e_1 \Rightarrow (A \rightarrow B) \qquad \Gamma \vdash e_2 \Leftarrow A}{\Gamma \vdash e_1\, e_2 \Rightarrow B}\ \textsf{Syn}{\rightarrow}\textsf{Elim}$$

- Chk→Intro:
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The subsumption rule:

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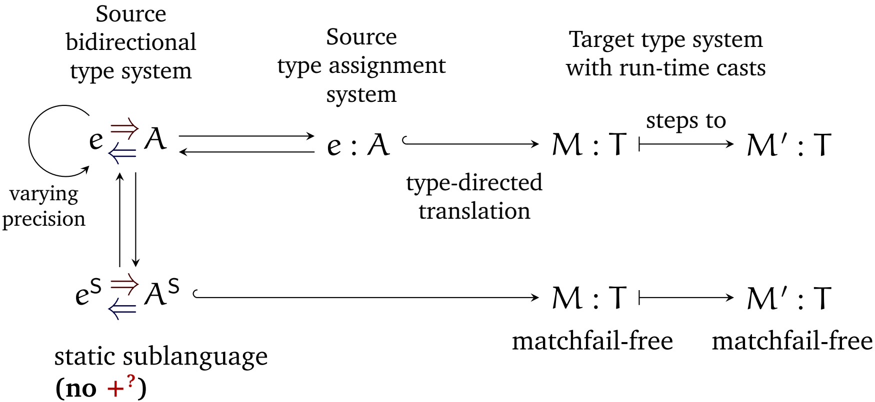
- Subformula property:
  Every type synthesized or checked flows from a type annotation.

Road map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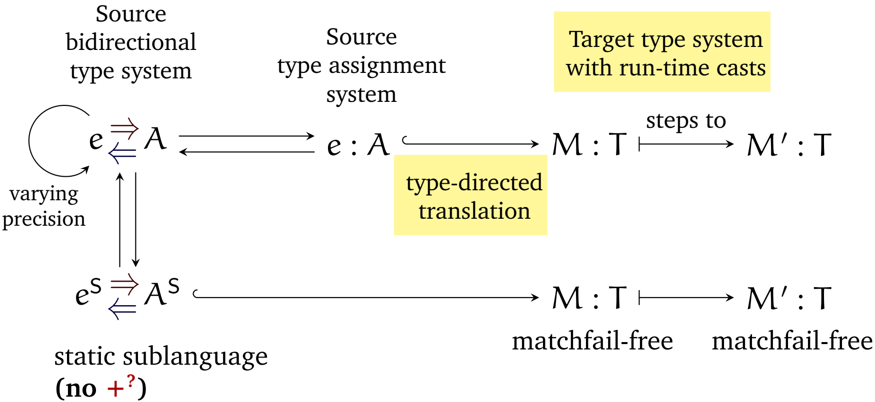
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Target language

- Target sum types include only **static** sums: $+$, $+_1$, $+_2$
- Casts between sums:

$$\langle +_1 \Leftarrow + \rangle(\mathtt{inj}_1\, v) \quad \text{will step to } \mathtt{inj}_1\, v$$

$$\langle +_2 \Leftarrow + \rangle(\mathtt{inj}_1\, v) \quad \text{will step to } \mathtt{matchfail}$$

Type-directed translation: add casts

Where **directed consistency** $\rightsquigarrow$ is used,
translation adds a cast from A' to B'

$$\frac{\Gamma \vdash e : A' \hookrightarrow M \qquad A' \rightsquigarrow B' \hookrightarrow \mathcal{C}}{\Gamma \vdash e : B' \hookrightarrow \mathcal{C}[M]}$$

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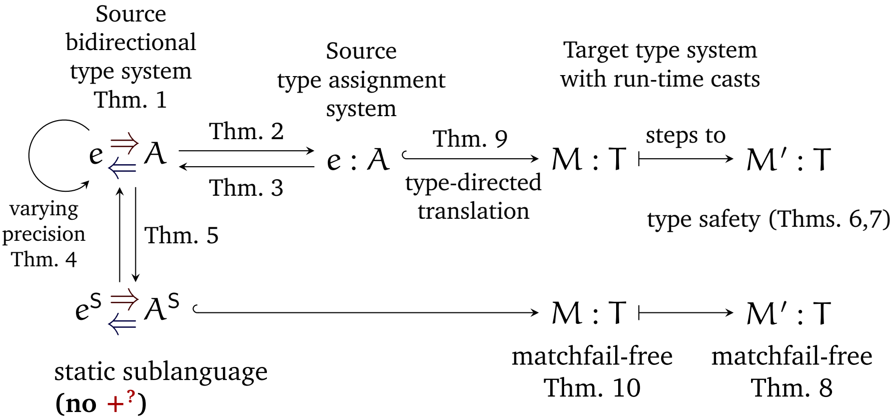
Metatheory



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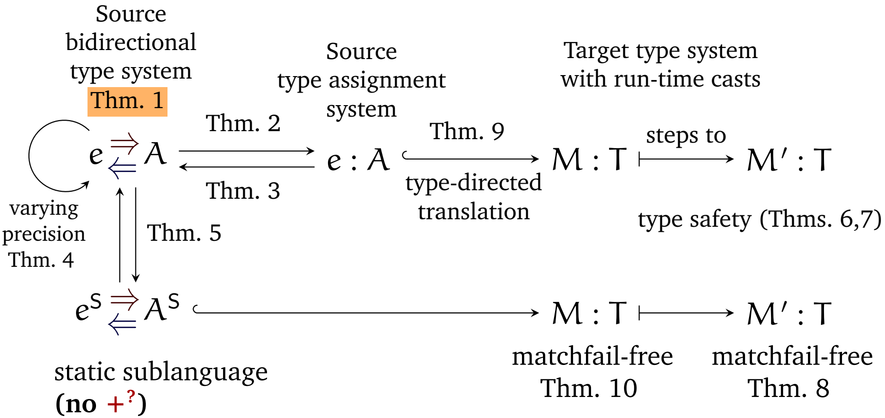

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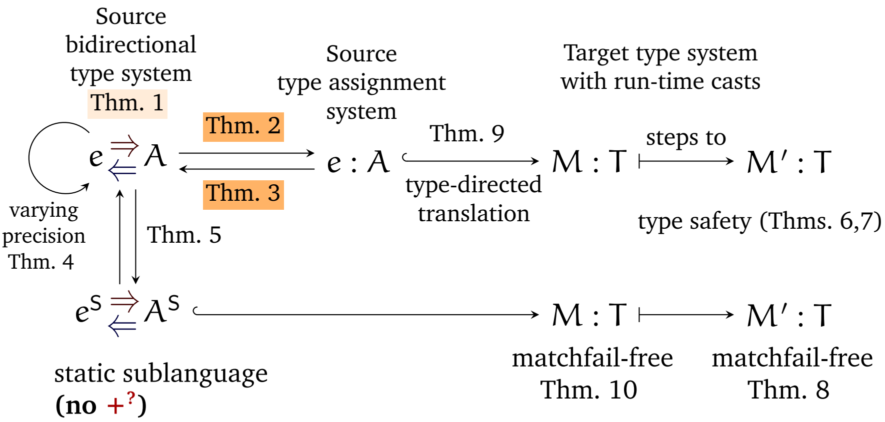
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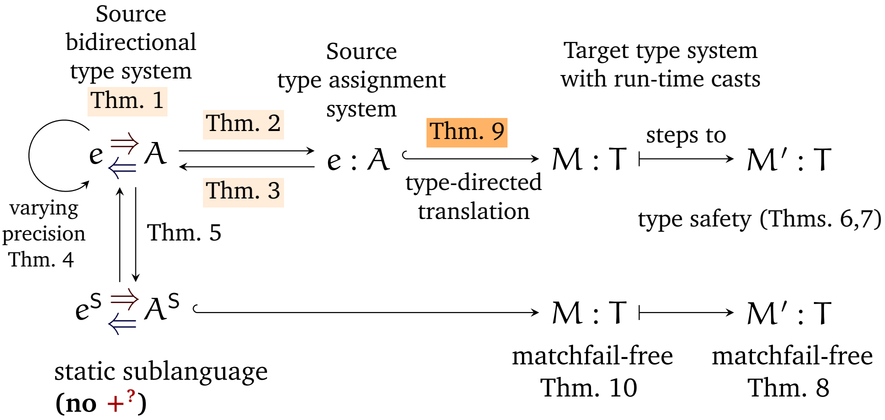
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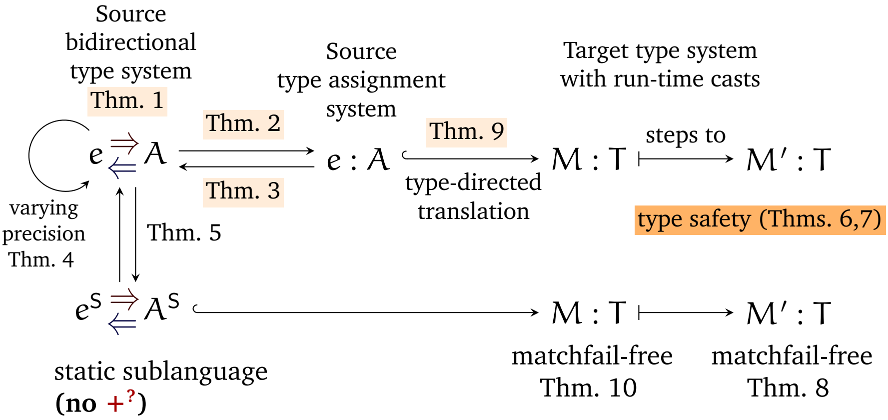

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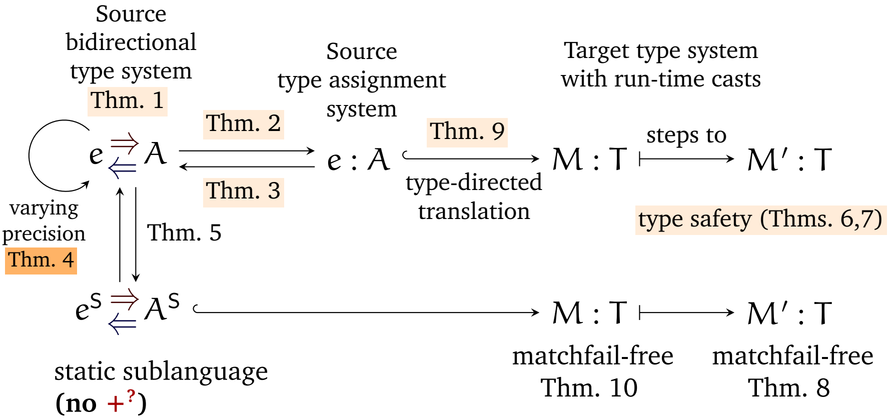
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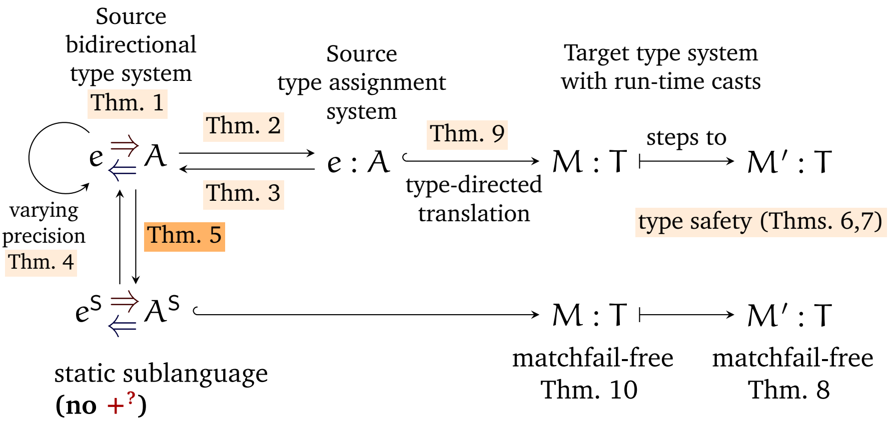
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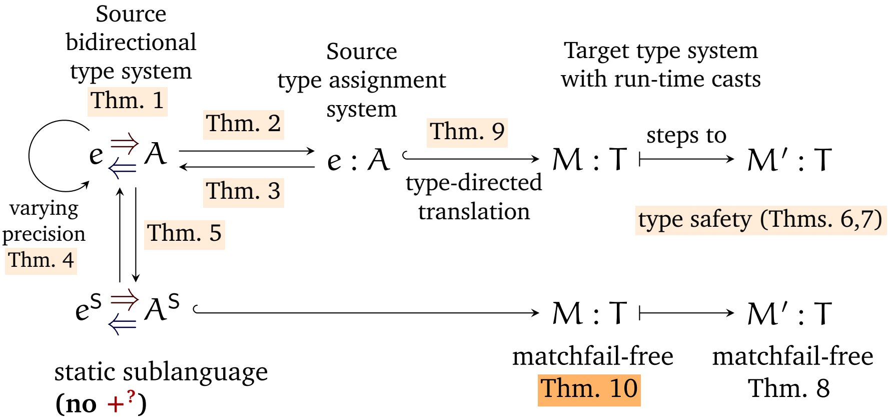
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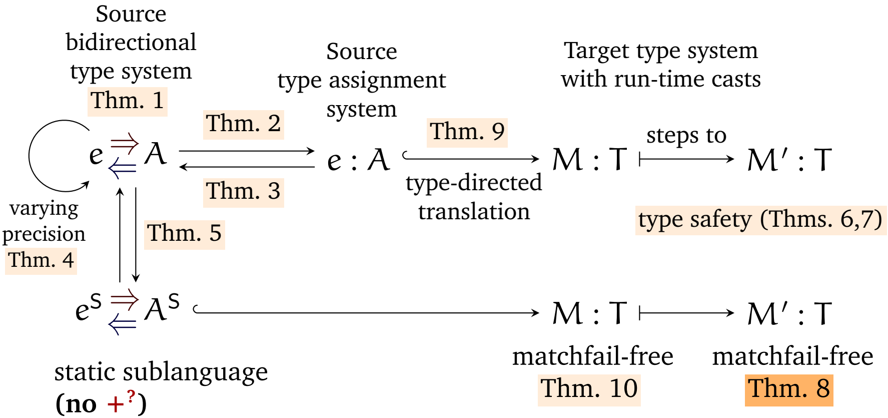
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Gradual guarantee (Siek et al. 2015)

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- Thm. 5: Static soundness and completeness
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Related work

Refinements:

- Datasort refinements:
  Freeman & Pfenning 1991, Davies 2005, …
  $A \sqsubset \tau$ says refinement (**sort**) A refines **type** τ.
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Gradual typing:

- Consistency (Siek and Taha 2006, ...)
- Consistent subtyping (Siek and Taha 2007, ...)
- Blame (Wadler & Findler 2009, ...)
- Subformula property (Garcia & Cimini 2015)

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- Add more types (intersection, μ, $\forall$)
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- Unify and generalize

  (1) classic gradual typing, and
  (2) gradual sums

  through a new type constructor, guided by ideas from **abstracting gradual typing** (Garcia et al. 2016)

Conclusion

- Guided by type-theoretic intuition,
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- The subformula property of bidirectional typing
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Paper and proofs: `arxiv.org/abs/1611.02392`

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