Supplementary material for “Bidirectional polymorphism through greed and unions”

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Appendix: Supplementary Material

This appendix contains definitions, proofs, and example derivations that didn’t fit in the paper. In line-by-line proofs, I put a $\mathcal{Z}$ next to the final result.

Lemma 9. *If* Ω *completes* Γ *then* $\mathsf{dom}([\Omega]\Gamma) \subseteq \mathsf{dom}(\Gamma)$.

Proof. By induction on Ω. Since Ω completes Γ, the contexts are the same modulo hints and existential variables that are declared in both but only solved in Ω. In the case when $\Omega = \Omega', \widehat{\alpha}{=}A$ and $\Gamma = \Gamma', \widehat{\alpha}$: from the definition, $[\Omega]\Gamma = [\Omega']([A/\widehat{\alpha}]\Gamma')$. By IH, $\mathsf{dom}([\Omega']([A/\widehat{\alpha}]\Gamma') \subseteq \mathsf{dom}([A/\widehat{\alpha}]\Gamma')$, and substituting for $\widehat{\alpha}$ in Γ' does not change its domain at all. ☐

Lemma 11. *Given a context* Ω *that completes* Γ*, if* $[\Omega]\Gamma \vdash [\Omega]A\ \mathsf{wf}$ *then* $\Gamma \vdash A\ \mathsf{wf}$.

Proof. To show $\Gamma \vdash A\ \mathsf{wf}$, we show $FV(A) \subseteq \mathsf{dom}(\Gamma)$.

For all $\widehat{\alpha}$ in $FV(A)$: Suppose $\widehat{\alpha} \notin \mathsf{dom}(\Gamma)$. By definition of completion, $\mathsf{dom}(\Gamma) = \mathsf{dom}(\Omega)$ so $\widehat{\alpha} \notin \mathsf{dom}(\Omega)$. Thus, applying Ω to A cannot substitute for $\widehat{\alpha}$, and $\widehat{\alpha} \in FV([\Omega]A)$. By definition of well-formedness, $FV([\Omega]A) \subseteq \mathsf{dom}([\Omega]\Gamma)$, which by Lemma 9 is $\subseteq \mathsf{dom}(\Gamma)$. Therefore $\widehat{\alpha} \in \mathsf{dom}(\Gamma)$, a contradiction. ☐

Lemma 12 (Well-Formedness)**.** *If* $\mathcal{D} :: \Gamma \vdash \ldots \dashv \Gamma'$ *then for any solved* $\widehat{\alpha} \in \mathsf{dom}(\Gamma)$*, it is the case that* $\Gamma = \Gamma_1, \widehat{\alpha}{=}A, \Gamma_2$ *and* $\Gamma_1 \vdash A\ \mathsf{wf}$*, and likewise for any solved* $\widehat{\alpha} \in \mathsf{dom}(\Gamma')$.

Proof. By induction on $\mathcal{D}$. In the 6 rules that introduce existential solutions, the well-formedness of the solution is either explicit ($\widehat{\alpha}^{=}\mathrm{L}\leqq$, $\widehat{\alpha}^{=}\mathrm{R}\leqq$) or is evident from the context ($\to\mathrm{I}\widehat{\alpha}$, $\to\mathrm{E}\widehat{\alpha}$, $\to\widehat{\alpha}\mathrm{L}\leqq$, $\to\widehat{\alpha}\mathrm{R}\leqq$). ☐

Definition 14 (Ordering of subtyping judgments)**.** Given $\mathcal{J}_1 = \Gamma_1 \vdash A_1 \leqq B_1 \dashv \ldots$ and $\mathcal{J}_2 = \Gamma_2 \vdash A_2 \leqq B_2 \dashv \ldots$, the order $\prec$ is defined lexicographically by

(1) the numbers of hints in Γ_1 and in Γ_2, under $<$;

(2) if $B_1 = B_2$ and $\Gamma_1 = \Gamma_2$, the angst of A_1 versus A_2; or, if $A_1 = A_2$ and $\Gamma_1 = \Gamma_2$, the angst of B_1 versus B_2;

(3) $\{A_1, B_1\} \prec \{A_2, B_2\}$;

(4) $A_1 = A_2$ and $B_1 = B_2$ where all existential variables in $A_1 (= A_2)$ are solved in Γ_1 but not in Γ_2; or, the same, swapping B_1 and B_2 for A_1 and A_2.

Definition 15 (Ordering of typing judgments)**.** Given $\mathcal{J}_1 = \Gamma_1 \vdash e_1 \Uparrow/\Downarrow C_1 \dashv \Gamma_1'$ and $\mathcal{J}_2 = \Gamma_2 \vdash e_2 \Uparrow/\Downarrow C_2 \dashv \Gamma_2'$, we define $\mathcal{J}_1 \preceq \mathcal{J}_2$ by the lexicographic ordering of:

(1) e_1 and e_2 (subterm ordering);

(2) the directions, considering $\Uparrow$ smaller than $\Downarrow$;

(3a) If both are checking judgments:

- (i) $C_1 \preceq C_2$;
- (ii) $\Gamma_1 = \Gamma_2$ and C_1 has less angst then C_2; or
- (iii) all existential variables in $C_1 (= C_2)$ are solved in Γ_1 but not in Γ_2

(3b) If both are synthesis judgments:

- (i) the number of hints in Γ_1' versus Γ_2'; if equal,
- (ii) $C_2 \preceq C_1$;
- (iii) C_2 has less angst with respect to Γ_2' than C_1 with respect to Γ_1'.

Theorem 16 (Decidability of Subtyping and Contextual Matching)**.** *Given Γ, A, and B, the existence of Γ' such that $\Gamma \vdash A \leqq B \dashv \Gamma'$ in System Bi$^{\widehat{\alpha}}$ is decidable.*

Moreover, given Γ_0, A_0 and Γ, the existence of A such that $(\Gamma \vdash A_0) \lesssim (\Gamma \vdash A)$ is decidable.

Proof. We show that the premises of each rule are smaller, under the defined partial order, than the conclusion. We also note that in each rule, we have enough information to apply the induction hypothesis for each premise.

$\forall$L-hint$\leqq$'s premise is smaller by part (1) of Definition 14.

In ExSubstL$\leqq$ and ExSubstR$\leqq$, use part (2).

In $\forall$L$\widehat{\alpha}\leqq$ (converting $\widehat{\alpha}$ to α), $\to\leqq$ and $\forall$R$\leqq$, use part (3).

In $\to\widehat{\alpha}$L$\leqq$ and $\to\widehat{\alpha}$R$\leqq$, use part (4).

The rules $\mathbf{1}\leqq$, αRefl$\leqq$, $\widehat{\alpha}$Refl$\leqq$, $\widehat{\alpha}^{=}$L$\leqq$, $\widehat{\alpha}^{=}$R$\leqq$ have no interesting premises.

For contextual matching, the rule empty-σ has no premises, while the length of Γ_0 is reduced by every other rule in Figure 6. ☐

Theorem 17 (Decidability of Typing)**.**

(i) Given Γ, e, and C, it is decidable whether there exists Γ' such that $\Gamma \vdash e \Downarrow C \dashv \Gamma'$.

(ii) Given Γ and e it is decidable whether there exist Γ' and C such that $\Gamma \vdash e \Uparrow C \dashv \Gamma'$.

Proof. We show that the premises of each rule are smaller, under the defined partial order, than the conclusion. We also note that in each rule, we have enough information to apply the induction hypothesis for each premise. For example, in $\to$E, we have $e = e_1 e_2$, giving us an e_1 for $\to$E's synthesizing premise; applying the i.h. there gives a type for the second, checking, premise.

var and $\mathbf{1}$I have no premises.

By part (1), the premises of anno, $\to$I, $\to$E, hint, $\to$E$\widehat{\alpha}$ have a smaller term than the conclusion.

sub's first premise is smaller by part (2); the second premise is decidable by Theorem 16.

$\forall$E-hint's premise is smaller by part (3b)(i). Contextual matching is decidable by Theorem 16.

$\forall$I's premise is smaller by part (3a)(i); $\forall$E$\widehat{\alpha}$'s premise is smaller by part (3b)(ii).

ExSubst$\Downarrow$'s premise is smaller by part (3a)(ii); ExSubst$\Uparrow$'s, by part (3b)(iii).

$\to$I$\widehat{\alpha}$'s premise is smaller by part (3a)(iii). ☐

Theorem 18 (Soundness of System Bi$^{\widehat{\alpha}}$). *If $\Gamma \vdash \mathcal{J} \dashv \Gamma'$ and Ω completes Γ' then $[\Omega]\Gamma' \vdash [\Omega]\mathcal{J}'$, where $\mathcal{J}'$ is $\mathcal{J}$ with any* **hint** $\ldots$ **in** e *subterms replaced by e and hints in annotations removed.*

Proof. Since Ω completes Γ', we have $\Omega \supseteq \Gamma'$: any variable $\widehat{\alpha}$ that is solved in Γ' is also solved, *and has the same solution*, in Ω. Moreover, it follows from Lemma 13 that $\Gamma' \supseteq \Gamma$. Since $\supseteq$ is a transitive relation, any $\widehat{\alpha}$ solved in Γ is solved and has the same solution in Ω.

When applying the IH, we must ensure that the Ω and Γ' we apply the IH with are in sync. For example, in the case for $\forall$I the output context in the subderivation is $\Gamma', \alpha, \Gamma_Z$ while the output context for the derivation is Γ'. The given Ω completes Γ', not $\Gamma', \alpha, \Gamma_Z$, so it must be extended as follows: Add solutions in Γ_Z to Ω; for unsolved variables $\widehat{\beta}$, choose *any* well-formed type B—$\mathbf{1}$ is the easiest choice since it has no free type variables and is thus well-formed in every context—and add $\widehat{\beta}{=}B$ to Ω. This works because $\forall$I strips out all the declarations in Γ_Z, so $\widehat{\beta}$ is about to leave this world unsolved, and therefore unconstrained.

In the $\forall$E$\widehat{\alpha}$ case, the IH gives $[\Omega]\Gamma \vdash e \Uparrow \forall\alpha.\,[\Omega]A$. Since Ω is solved, $\widehat{\alpha}{=}A' \in \Omega$, and by Lemma 12, $\Gamma \vdash A'$ wf. By Corollary 10, $[\Omega]\Gamma \vdash [\Omega]A'$ wf. By $\forall$E, $[\Omega]\Gamma \vdash e \Uparrow [[\Omega]A'/\alpha]([\Omega]A)$. By a property of substitutions, $[[\Omega]A'/\alpha]([\Omega]A) = [\Omega][A'/\alpha]A$, giving the result.

In the ExSubst$\Downarrow$ case, the IH yields $[\Omega]\Gamma \vdash e \Downarrow [\Omega]\Gamma(\widehat{\alpha})$; the variable $\widehat{\alpha}$ cannot be free in $\Gamma(\widehat{\alpha})$, and we earlier noted that $\Omega(\widehat{\alpha}) = \Gamma(\widehat{\alpha})$, so in fact $[\Omega]\Gamma(\widehat{\alpha}) = [\Omega]\Omega(\widehat{\alpha}) = [\Omega]\widehat{\alpha}$, giving the result. ExSubst$\Uparrow$ and ExSubst$\{$L,R$\}\leq$ are similar.

In the $\to$I$\widehat{\alpha}$ case, the IH gives $[\Omega]\Gamma, x{:}([\Omega]\widehat{\alpha_1}) \vdash e_0 \Downarrow [\Omega]\widehat{\alpha_2}$. By $\to$I, $[\Omega]\Gamma \vdash \lambda x.\,e_0 \Downarrow ([\Omega]\widehat{\alpha_1}) \to ([\Omega]\widehat{\alpha_2})$. The declaration $\widehat{\alpha}{=}\widehat{\alpha_1} \to \widehat{\alpha_2}$ is in Γ, so by Lemma 13 it is also in Ω. Thus, we have $\ldots \Downarrow [\Omega]\widehat{\alpha}$, which was to be shown.

In the $\widehat{\alpha}^{=}$L$\leq$ case, we have $(\widehat{\alpha}{=}B) \in \Gamma'$. want $[\Omega]\Gamma \vdash [\Omega]\widehat{\alpha} \leq [\Omega]B$. By Lemma 13, $(\widehat{\alpha}{=}B) \in \Omega$, so $[\Omega]\widehat{\alpha} = [\Omega]B$. The result follows by reflexivity of $\leqq$. The $\widehat{\alpha}^{=}$R$\leq$ case is symmetric.

The $\to\widehat{\alpha}$L$\leq$, $\to\widehat{\alpha}$R$\leq$ cases use similar reasoning as the $\to$I$\widehat{\alpha}$ case.

The remaining cases are straightforward. ∎

$$
\dfrac{
\begin{array}{c}\dfrac{\Gamma \vdash e \Uparrow \forall\alpha.\,\alpha\to\alpha}{\Gamma \vdash e \Uparrow [\mathbf{1}\to\mathsf{int}]\,(\alpha\to\alpha)}\,\forall\mathrm{E}\\ \vdots\end{array}
\qquad
\begin{array}{c}\dfrac{\Gamma, x{:}\mathbf{1} \vdash \ldots \Downarrow \mathsf{int}}{\Gamma \vdash \lambda x.\,\ldots \Downarrow \mathbf{1}\to\mathsf{int}}\,{\to}\mathrm{I}\\ \vdots\end{array}
\qquad
\begin{array}{c}\mathbf{1}\to\mathsf{int} \leqq \mathbf{1}\to\mathsf{int}\\ \vdots\end{array}
}{\cdots}
$$

$$
\dfrac{
\begin{array}{c}\dfrac{\Gamma \vdash e \Uparrow \forall\alpha.\,\alpha\to\alpha \dashv \Gamma}{\Gamma \vdash e \Uparrow [\widehat{\alpha}/\alpha]\,(\alpha\to\alpha) \dashv \Gamma, \widehat{\alpha}}\,\forall\mathrm{E}\widehat{\alpha}\\ \vdots\end{array}
\qquad
\begin{array}{c}\dfrac{\Gamma_2, x{:}\widehat{\alpha_1} \vdash \ldots \Downarrow \widehat{\alpha_2} \dashv \Gamma_2, x{:}\widehat{\alpha_1}}{\Gamma, \widehat{\alpha} \vdash \lambda x.\,\ldots \Downarrow \widehat{\alpha} \dashv \Gamma_2}\,{\to}\mathrm{I}\widehat{\alpha}\\ \vdots\end{array}
\qquad
\begin{array}{c}\Gamma_2 \vdash \widehat{\alpha} \leqq \mathbf{1}\to\mathsf{int} \dashv \overbrace{\Gamma, \widehat{\alpha_1}{=}\mathbf{1}, \widehat{\alpha_2}{=}\mathsf{int}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}}^{\Omega}\\ \vdots\end{array}
}{\cdots}
$$

Figure 14: Corresponding derivations in System Bi (above) and System Bi$^{\widehat{\alpha}}$ (below)

Stipulating that certain occurrences of $\mathbf{1} \to \mathsf{int}$ in the middle and right of the derivation do in fact flow from the occurrence of $\mathbf{1} \to \mathsf{int}$ on the left, the System Bi$^{\widehat{\alpha}}$ derivation should look like the one at the bottom of Figure 14, where $\Gamma_2 = \Gamma, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}$. For the various judgments $\Gamma_1' \vdash \ldots \dashv \Gamma_2'$ in the System Bi$^{\widehat{\alpha}}$ derivation, the contexts Γ_1' and Γ_2' don't disagree with Ω; they may say less—for example, just after we create $\widehat{\alpha}$ on the left there is no information about $\widehat{\alpha}$—but they don't contradict it.

Theorem 21 (Predicative Completeness). *For any Ω and Γ_1' and predicative derivation $\mathcal{D} :: \Gamma \vdash [\Omega]\mathcal{J}$ in System Bi, provided that*

(1) Ω *is predicative (for any* $\widehat{\alpha}$*, the type* $\Omega(\widehat{\alpha})$ *is monomorphic) and articulated*

(2) Ω *completes* Γ_1'*, and* $[\Omega]\Gamma_1' = \Gamma$

$$
\begin{array}{llll}
\textit{then} & [\Omega]\Gamma_1' \vdash [\Omega]A' \leq [\Omega]B' & \Longrightarrow & \Gamma_1' \vdash A' \leqq B' \dashv \Gamma_2' \\
& [\Omega]\Gamma_1' \vdash e \Downarrow [\Omega]A' & \Longrightarrow & \Gamma_1' \vdash e \Downarrow A' \dashv \Gamma_2' \\
& [\Omega]\Gamma_1' \vdash e \Uparrow C & \Longrightarrow & \Gamma_1' \vdash e \Uparrow C' \dashv \Gamma_2' \\
& & & \textit{for some } C' \textit{ such that} \\
& & & C = [\Omega]C'
\end{array}
$$

Proof. By induction on $\mathcal{D}$.

Assuming the given types $[\Omega]A'$, etc. are well-formed, by Lemma 11 the types A', etc. are well-formed under Γ_1'. But the type C in the synthesis judgment is well-formed under Γ, while the type C' in the consequent of the theorem is well-formed under Γ_2'—and not necessarily under Γ_1', as Γ_2' may contain existential type variables that Γ_1' does not.

- **Case** $\rightarrow\leqq$: $\boxed{\mathcal{D} :: \dfrac{\Gamma \vdash B_1 \leqq A_1 \qquad \Gamma \vdash A_2 \leqq B_2}{\Gamma \vdash \underbrace{A_1 \rightarrow A_2}_{[\Omega]A'} \leqq \underbrace{B_1 \rightarrow B_2}_{[\Omega]B'}}}$

  We know that $[\Omega]A' = A_1 \rightarrow A_2$. Either $\{\rightarrow A' \text{ case}\}$ $A' = A_1' \rightarrow A_2'$ (so $[\Omega]A' = [\Omega]A_1' \rightarrow [\Omega]A_2' = A_1 \rightarrow A_2$) or $\{\widehat{\alpha}A' \text{ case}\}$ $A' = \widehat{\alpha}$ (so $[\Omega]A' = [\Omega]\widehat{\alpha}$). Similarly, we distinguish $\{\rightarrow B' \text{ case}\}$ and $\{\widehat{\beta}B' \text{ case}\}$ depending on whether B' is $B_1' \rightarrow B_2'$ or $\widehat{\beta}$. (Note that possibly $\widehat{\beta} = \widehat{\alpha}$.)

  – $\{\rightarrow A'$ and $\rightarrow B'$ case$\}$:

$\Gamma_1' \vdash B_1' \leqq A_1' \dashv \Gamma_2'$	By IH
$\Gamma_2' \vdash A_2' \leqq B_2' \dashv \Gamma_3'$	By IH
$\Gamma_1' \vdash A_1' \rightarrow A_2' \leqq B_1' \rightarrow B_2' \dashv \Gamma_3'$	By $\rightarrow\leqq$

  – $\{\widehat{\alpha}A'$ and $\rightarrow B'$ case$\}$:

$\Gamma_1' \vdash A_1' \rightarrow A_2' \leqq B_1' \rightarrow B_2' \dashv \Gamma_3'$	As preceding case

  If Γ_1' includes a solution for $\widehat{\alpha}$, then:

  ☞ $\Gamma_1' \vdash \widehat{\alpha} \leqq B_1' \rightarrow B_2' \dashv \Gamma_3'$ By ExSubstL$\leqq$

  Otherwise, Γ_1' does not include a solution for $\widehat{\alpha}$.

  * $\Omega(\widehat{\alpha}) = [\Omega]A' = A_1 \rightarrow A_2$ must have the form $\widehat{\alpha_1} \rightarrow \widehat{\alpha_2}$, because Ω is predicative and articulated. We assumed that Γ_1' does not include a solution for $\widehat{\alpha}$, so $\Gamma_1' = \Gamma_L, \widehat{\alpha}, \Gamma_R$. Let $\Gamma_+ = \Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\rightarrow}\widehat{\alpha_2}, \Gamma_R$.

	$\Gamma_+ \vdash B_1' \leqq \widehat{\alpha_1} \dashv \Gamma_M$	By IH on $\Gamma \vdash B_1 \leqq A_1$, taking $\Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\rightarrow}\widehat{\alpha_2}$ as Γ_1'
	$\Gamma_M \vdash \widehat{\alpha_2} \leqq B_2' \dashv \Gamma_2'$	By IH
	$\Gamma_+ \vdash \widehat{\alpha_1} \rightarrow \widehat{\alpha_2} \leqq B_1' \rightarrow B_2' \dashv \Gamma_2'$	By $\rightarrow\leqq$
	$\Gamma_+ \vdash \widehat{\alpha} \leqq B_1' \rightarrow B_2' \dashv \Gamma_2'$	By ExSubstL$\leqq$
☞	$\Gamma_1' \vdash \widehat{\alpha} \leqq B_1' \rightarrow B_2' \dashv \Gamma_2'$	By $\rightarrow\widehat{\alpha}L\leqq$

  – $\{\rightarrow A'$ and $\widehat{\beta}B'$ case$\}$: Symmetric to the $\{\widehat{\alpha}A'$ and $\rightarrow B'$ case$\}$.

  – $\{\widehat{\alpha}A'$ and $\widehat{\beta}B'$ case$\}$: If either $\widehat{\alpha}$ or $\widehat{\beta}$ is solved in Γ_1', then the solution in Γ_1' has an $\rightarrow$ at its head (since the solution in Ω does). Using suitably articulated contexts, use the IH, then use ExSubst and $\rightarrow\widehat{\alpha}L\leqq$ or $\rightarrow\widehat{\alpha}R\leqq$ as needed.

  If neither is solved and $\widehat{\alpha} = \widehat{\beta}$, then the result follows by $\widehat{\alpha}$Refl$\leqq$.

Otherwise, neither is solved and $\widehat{\alpha} \neq \widehat{\beta}$. So add a solution for whichever of $\widehat{\alpha}$ and $\widehat{\beta}$ is declared last in Γ_1'. Suppose without loss of generality that $\Gamma_1' = \Gamma_L, \widehat{\alpha}, \Gamma_C, \widehat{\beta}, \Gamma_R$.

$\Gamma_1' \vdash \widehat{\alpha} \leqq \widehat{\beta} \dashv \Gamma_L, \widehat{\alpha}, \Gamma_C, \widehat{\beta}{=}\widehat{\alpha}$ By $\widehat{\alpha}^{=}$R$\leqq$

- **Case** αRefl$\leqq$: $\boxed{\mathcal{D} :: \quad \dfrac{}{\Gamma \vdash \alpha \leqq \alpha}}$

  We have $\alpha = [\Omega]A' = [\Omega]B'$. The types A' and B' can each be α or various existential variables.

  If $A' = B' = \alpha$, the result follows by αRefl$\leqq$, giving $\Gamma_1' \vdash \alpha \leqq \alpha \dashv \Gamma_1'$.

  If $A' = \alpha$ and B' is some solved $\widehat{\beta}$, the result follows by αRefl$\leqq$, yielding $\Gamma_1' \vdash \alpha \leqq \alpha \vdash \Gamma_1'$ then ExSubstR$\leqq$ for $\Gamma_1' \vdash \alpha \leqq \widehat{\beta} \dashv \Gamma_1'$.

  If $\widehat{\beta}$ is unsolved: $\widehat{\beta}$ is well-formed in Γ_1', so $\Gamma_1' = \Gamma_L, \widehat{\beta}, \Gamma_R$. Applying $\widehat{\alpha}^{=}$R$\leqq$ gives $\Gamma_L, \widehat{\beta}, \Gamma_R \vdash \alpha \leqq \widehat{\beta} \dashv \Gamma_L, \widehat{\beta}{=}\alpha, \Gamma_R$. Let $\Gamma_2' = \Gamma_L, \widehat{\beta}{=}\alpha, \Gamma_R$. Substituting gives $\Gamma_1' \vdash \alpha \leqq \widehat{\beta} \dashv \Gamma_2'$, which was to be shown.

  The subcases where $B' = \alpha$ and A' is some solved $\widehat{\beta}$ are symmetric to the last two.

  If $A' = \widehat{\gamma}$ and $B' = \widehat{\beta}$, first apply αRefl$\leqq$, then:

  - If both are solved in Γ_1', apply ExSubstL$\leqq$ then ExSubstR$\leqq$.
  - If only $\widehat{\gamma}$ is solved, apply ExSubstL$\leqq$ then $\widehat{\alpha}^{=}$R$\leqq$.
  - If only $\widehat{\beta}$ is solved, apply ExSubstR$\leqq$ then $\widehat{\alpha}^{=}$L$\leqq$ (symmetric to the last).
  - If neither is solved: Both $\widehat{\gamma}$ and $\widehat{\beta}$ are well-formed under Γ_1'. Either $\widehat{\gamma}$ comes first or $\widehat{\beta}$ comes first. Suppose $\widehat{\beta}$ comes first. Then $\widehat{\alpha}^{=}$L$\leqq$ gives $\Gamma_1' \vdash \widehat{\gamma} \leqq \widehat{\beta} \dashv \ldots, \widehat{\alpha}{=}\widehat{\beta}, \ldots$.

- **Case 1**$\leqq$: Similar to the previous case, using **1**$\leqq$ in place of αRefl$\leqq$.

- **Case** $\forall$L$\leqq$: $\boxed{\mathcal{D} :: \quad \dfrac{\Gamma \vdash [C/\alpha]A_0 \leqq B \qquad \Gamma \vdash C \text{ wf}}{\Gamma \vdash \underbrace{\forall\alpha.\, A_0}_{[\Omega]A'} \leqq \underbrace{B}_{[\Omega]B'}}}$

  We know that $[\Omega]A' = \forall\alpha.\, A_0$. Either $\{\forall A' \text{ case}\}$ $A' = \forall\alpha.\, A_0'$, so $[\Omega]A' = \forall\alpha.\, [\Omega]A_0'$, or $\{\widehat{\gamma}A \text{ case}\}$ $A' = \widehat{\gamma}$ so $[\Omega]\widehat{\gamma} = \forall\alpha.\, \ldots$, which is impossible by the assumption that Ω is predicative.

  - $\{\forall A' \text{ case}\}$:
    Choose a fresh $\widehat{\alpha}$. Let $\Omega' = \Omega, \mathit{Artic}(\widehat{\alpha}{=}C)$.

A_0	$=$	$[\Omega]A_0'$	Above
$[C/\alpha]A_0$	$=$	$[C/\alpha][\Omega]A_0'$	Applying $[C/\alpha]$ to both sides
	$=$	$[\Omega]([C/\alpha]A_0')$	Permutation (no ex. vars. in C)
	$=$	$[\Omega]([C/\widehat{\alpha}][\widehat{\alpha}/\alpha]A_0')$	$\widehat{\alpha}$ fresh
	$=$	$[\Omega, \mathit{Artic}(\widehat{\alpha}{=}C)][\widehat{\alpha}/\alpha]A_0'$	Definitions of articulation and substitution
	$=$	$[\Omega'][\widehat{\alpha}/\alpha]A_0'$	Definition of Ω' above

    Therefore $[C/\alpha]A_0 = [\Omega']([\widehat{\alpha}/\alpha]A_0')$, and we can apply the IH:

	$\Gamma_1', \widehat{\alpha} \vdash [\widehat{\alpha}/\alpha]A_0' \leqq B' \dashv \Gamma_R$	By IH with Ω'
	$\Gamma_R = \Gamma_2', \widehat{\alpha}[\ldots], \Gamma_Z$	$[\widehat{\alpha}/\alpha]A_0'$ well-formed under Γ_R, so $\widehat{\alpha} \in \mathsf{dom}(\Gamma_R)$
☞	$\Gamma_1' \vdash \forall\alpha.\, A' \leqq B' \dashv \Gamma_2'$	By $\forall$L$\widehat{\alpha}\leqq$

- **Case** $\forall R\leqq$: $\boxed{\mathcal{D} :: \dfrac{\Gamma, \beta \vdash A \leqq B_0}{\Gamma \vdash \underbrace{A}_{[\Omega]A'} \leqq \underbrace{\forall\beta.\, B_0}_{[\Omega]B'}}}$

  We know that $[\Omega]B' = \forall\beta.\, B_0$. Either $\{\forall B'$ case$\}$ $B' = \forall\beta.\, B_0'$ (so $[\Omega]B' = \forall\beta.\, [\Omega]B_0'$) or $\{\widehat{\gamma}B$ case$\}$ $B' = \widehat{\gamma}$.

  – $\{\forall B'$ case$\}$:

  $$\begin{array}{rll} & \Gamma_1', \beta \vdash A' \leqq B' \dashv \Gamma_2'' & \text{By IH} \\ & \Gamma_2'' = \Gamma_2', \beta, \Gamma_Z & \text{By } \overline{\Gamma_2''} = \Gamma \text{ (follows from Lemma 13)} \\ ☞ & \Gamma_1' \vdash A' \leqq \forall\beta.\, B_1' \dashv \Gamma_2' & \text{By } \forall R\leqq \end{array}$$

  – $\{\widehat{\gamma}B'$ case$\}$:

  Applying Ω to $B' = \widehat{\gamma}$ gives $[\Omega]B' = [\Omega]\widehat{\gamma}$, which is equal to $\Omega(\widehat{\gamma})$. But since $[\Omega]B' = \forall\beta.\, B_0$, we have $\Omega(\widehat{\gamma}) = \forall\beta.\, B_0$, which contradicts our assumption that Ω is predicative: this case is impossible.

- **Case** var: $\boxed{\mathcal{D} :: \dfrac{\Gamma(x) = A}{\Gamma \vdash x \Uparrow A}}$

  $\Gamma = [\Omega]\Gamma_1'$. Therefore $\Gamma(x) = [\Omega](\Gamma_1'(x))$. So $\Gamma_1'(x) = A'$ where $[\Omega]A' = A$. The result, $\Gamma_1' \vdash x \Uparrow A' \dashv \Gamma_1'$, follows by var.

- **Case** sub: $\boxed{\mathcal{D} :: \dfrac{\Gamma \vdash e \Uparrow B \qquad \Gamma \vdash B \leqq A}{\Gamma \vdash e \Downarrow A}}$

  By IH, $\Gamma_1' \vdash e \Uparrow B' \dashv \Gamma_M$ where $[\Omega]B' = B$. We have $[\Omega]A' = A$. By IH, $\Gamma_M \vdash B' \leqq A' \dashv \Gamma_2'$. The result follows by sub.

- **Case** anno: $\boxed{\mathcal{D} :: \dfrac{N \lesssim (\Gamma \vdash A) \qquad \Gamma \vdash e \Downarrow A}{\Gamma \vdash (e : N) \Uparrow A}}$

  The result follows by the IH and anno. (The $\lesssim$ premise of anno in System $\mathrm{Bi}^{\widehat{\alpha}}$ does not involve existential contexts; see Section 3.2.1.)

- **Case** $\to$I: $\boxed{\mathcal{D} :: \dfrac{\Gamma, x{:}A_1 \vdash e \Downarrow A_2}{\Gamma \vdash \lambda x.\, e \Downarrow \underbrace{A_1 \to A_2}_{[\Omega]A'}}}$

  If $A' = A_1' \to A_2'$ (with $[\Omega]A_1' = A_1$ and $[\Omega]A_2' = A_2$): The IH gives $\Gamma_1', x{:}A_1' \vdash e \Downarrow A_2' \dashv \Gamma_M$. By Lemma 5, $\overline{\Gamma_M} = \overline{\Gamma_1'}$; then, by Lemma 6, $\Gamma_M = \Gamma_2', x{:}A_1', \Gamma_R$. Applying $\to$I gives $\Gamma_1' \vdash \lambda x.\, e \Downarrow A_1' \to A_2' \dashv \Gamma_2'$, which was to be shown.

  Otherwise, $A' = \widehat{\alpha}$ and $\Omega(\widehat{\alpha}) = \widehat{\alpha_1} \to \widehat{\alpha_2}$, where $A_1 = [\Omega]\widehat{\alpha_1}$ and $A_2 = [\Omega]\widehat{\alpha_2}$.

  – $\{$solved case$\}$: $\widehat{\alpha}$ solved in Γ_1'; since Γ_1' is articulated, $\widehat{\alpha}{=}\widehat{\alpha_1} \to \widehat{\alpha_2} \in \Gamma_1'$.

  $$\begin{array}{rll} & \Gamma_1', x{:}\widehat{\alpha_1} \vdash e \Downarrow \widehat{\alpha_2} \dashv \Gamma_2', x{:}\widehat{\alpha_1}, \Gamma_R & \text{By IH} \\ & \Gamma_1' \vdash \lambda x.\, e \Downarrow \widehat{\alpha_1} \to \widehat{\alpha_2} \dashv \Gamma_2' & \text{By } \to\text{I} \\ ☞ & \Gamma_1' \vdash \lambda x.\, e \Downarrow \widehat{\alpha} \dashv \Gamma_2' & \text{By ExSubst}\Downarrow \end{array}$$

- {not-solved case}: $\widehat{\alpha}$ not solved in Γ_1': decompose Γ_1' into $\Gamma_{11}, \widehat{\alpha}, \Gamma_{12}$.

$$\begin{array}{rll}
\Gamma_{11}, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_{12}, x{:}\widehat{\alpha_1} \vdash e \Leftarrow \widehat{\alpha_2} \dashv \Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_{12}, x{:}\widehat{\alpha_1}, \Gamma_R & & \text{By IH} \\
\Gamma_{11}, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_{12} \vdash \lambda x.\, e \Leftarrow \widehat{\alpha_1} \to \widehat{\alpha_2} \dashv \Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_{12} & & \text{By} \to\!\text{I} \\
\Gamma_{11}, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_{12} \vdash \lambda x.\, e \Leftarrow \widehat{\alpha} \dashv \Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_{12} & & \text{By ExSubst}\Leftarrow \\
☞ \quad \Gamma_{11}, \widehat{\alpha}, \Gamma_{12} \vdash \lambda x.\, e \Leftarrow \widehat{\alpha} \dashv \Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2} & & \text{By} \to\!\text{I}\widehat{\alpha}
\end{array}$$

- **Case** $\to$E: $\boxed{\mathcal{D} :: \dfrac{\Gamma \vdash e_1 \Uparrow B \to A \qquad \Gamma \vdash e_2 \Leftarrow B}{\Gamma \vdash e_1\, e_2 \Uparrow \underbrace{A}_{[\Omega]A'}}}$

  By IH, $\Gamma_1' \vdash e_1 \Uparrow C' \dashv \Gamma_M$ where $[\Omega]C' = B \to A$.

  If $C' = B' \to A'$ then $[\Omega]B' = B$ and $[\Omega]A' = A$. By IH, $\Gamma_M \vdash e_2 \Leftarrow B' \dashv \Gamma_2'$. The result is by $\to$E.

  Otherwise, $C' = \widehat{\alpha}$ and $\Omega(\widehat{\alpha}) = \widehat{\alpha_1} \to \widehat{\alpha_2}$. Since $[\Omega]C' = B \to A$, we have $[\Omega]\widehat{\alpha_1} = B$ and $[\Omega]\widehat{\alpha_2} = A$. The type C' must be well-formed under Γ_1' and under Γ_M, so $\widehat{\alpha}$ must be defined within those contexts:

$$\Gamma_1' = \Gamma_{11}, \widehat{\alpha}, \Gamma_{12} \quad \text{and} \quad \Gamma_M = \Gamma_L, \widehat{\alpha}, \Gamma_R$$

  Therefore the IH really gave us $\Gamma_{11}, \widehat{\alpha}, \Gamma_{12} \vdash e_1 \Uparrow \widehat{\alpha} \dashv \Gamma_L, \widehat{\alpha}, \Gamma_R$. Applying the IH to $\Gamma \vdash e_2 \Leftarrow B$, with input context $\Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_R$ yields

$$\Gamma_L, \widehat{\alpha_1}, \widehat{\alpha_2}, \widehat{\alpha}{=}\widehat{\alpha_1}{\to}\widehat{\alpha_2}, \Gamma_R \vdash e_2 \Leftarrow \widehat{\alpha_1} \dashv \Gamma_2'$$

  $\to$E$\widehat{\alpha}$ gives $\Gamma_{11}, \widehat{\alpha}, \Gamma_{12} \vdash e_1 e_2 \Uparrow \widehat{\alpha_2} \dashv \Gamma_2'$, which is the same as $\Gamma_1' \vdash e_1 e_2 \Uparrow \widehat{\alpha_2} \dashv \Gamma_2'$, which was to be shown.

- **Case** $\mathbf{1}$I: Since $A = \mathbf{1}$, either $A' = \mathbf{1}$ and we just apply $\mathbf{1}$I, or $A' = \widehat{\alpha}$ where $[\Omega]\widehat{\alpha} = \mathbf{1}$, in which case the result follows by $\mathbf{1}$I and ExSubst$\Leftarrow$.

- **Case** $\forall$I: $\boxed{\mathcal{D} :: \dfrac{\Gamma, \alpha \vdash e \Leftarrow A_0}{\Gamma \vdash e \Leftarrow \underbrace{\forall\alpha.\, A_0}_{[\Omega]A'}}}$

  A' is either $\forall\alpha.\, A_0'$ or $\widehat{\beta}$. But if $A' = \widehat{\beta}$ then $[\Omega]\widehat{\beta} = \forall\alpha.\, A_0$, violating the assumption that Ω is predicative. Therefore $A' = \forall\alpha.\, A_0'$, and $[\Omega]A_0' = A_0$.

$$\begin{array}{rll}
\Gamma_1', \alpha \vdash e \Leftarrow A_0' \dashv \Gamma_2', \alpha, \Gamma_Z & & \text{By IH} \\
☞ \quad \Gamma_1' \vdash e \Leftarrow \forall\alpha.\, A_0' \dashv \Gamma_2' & & \text{By } \forall\text{I}
\end{array}$$

- **Case** $\forall$E: $\boxed{\mathcal{D} :: \dfrac{\Gamma \vdash e \Uparrow \forall\alpha.\, A_0 \qquad \Gamma \vdash B \text{ wf}}{\Gamma \vdash e \Uparrow [B/\alpha]A_0}}$

  Extend Ω with the articulation of $\widehat{\alpha}{=}B$, yielding Ω'. By IH, $\Gamma_1' \vdash e \Uparrow A' \dashv \Gamma_2'$ where $[\Omega']A' = \forall\alpha.\, A_0$. Since Ω is predicative, A' must have the form $\forall\alpha.\, A_0'$ where $[\Omega]A_0' = A_0$. By $\forall$E$\widehat{\alpha}$,

$$\Gamma_1' \vdash e \Uparrow [\widehat{\alpha}/\alpha]A_0' \dashv \Gamma_2', \widehat{\alpha}$$

  The context Ω' includes the articulation of $\widehat{\alpha}{=}B$, so $[\Omega]\widehat{\alpha} = B$. Then $[\Omega][\widehat{\alpha}/\alpha]A_0' = [B/\alpha]A_0$. □